Field theoretic models

on covariant quantum spaces

Habilitationsschrift

Harold Steinacker

Ludwig-Maximilians Universität
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Contents

Introduction

The topic of this thesis is the study of field theory on a particular class of quantum spaces with quantum group symmetries. These spaces turn out to describe certain D-branes on group manifolds, which were found in string theory.

It is an old idea going back to Heisenberg [1] that the ultraviolet (UV)–divergences of quantum field theory (QFT) are a reflection of our poor understanding of the physics of spacetime at very short distances, and should disappear if a quantum structure of spacetime is taken into account. One expects indeed that space-time does not behave like a classical manifold at or below the Planck scale, where quantum gravity should modify its structure. However, the scale where a quantum structure of spacetime becomes important might as well be much larger than the Planck scale. There has been considerable effort to study field theory on various non-commutative spaces, sparked by the development of noncommutative geometry [2]. With very little experimental guidance, finding the correct description is of course very difficult.

In recent years, quantized spaces have attracted a lot of attention in the framework of string-theory [3, 4, 5, 6, 7, 8, 9], for a somewhat different reason. Even though this thesis is not about string theory, we want to take advantage of this connection, and therefore we need to explain it briefly. This is independent of whether or not string theory is a theory of nature: it is certainly a rich mathematical laboratory, and provides new insights into physical aspects of noncommutative field theories. The connection with noncommutative spaces is through D-branes, which are submanifolds of the target space, on which open strings end. It turns out that these D-branes become non-commutative (=quantized) spaces if there is a non-vanishing B-field, i.e. a particular 2-form field in the target space. The important point is that these D-branes carry certain induced field theories, arising from the open strings which end on them. This discovery gave a boost to the study of field theories on such non-commutative spaces, and led to important new insights. Their realizations in string theory is strong support for the existence of "physically interesting" field theories on quantized spaces. In general, it is far from trivial to study QFT on noncommutative spaces, and results which originated from string theory such as the Seiberg-Witten map were very helpful to establish a certain intuitive understanding.

The space which is usually considered in this context is the so-called quantum plane $\mathbb{R}^n_\theta$, defined by the algebra of coordinate "functions"

$$[x_i, x_j] = i\theta_{ij}. \tag{0-1}$$

Here θ_{ij} is a constant antisymmetric tensor, which is related to the background B field mentioned

above. Field theory has been studied in considerable detail on this space, and a formulation of gauge theory was achieved using the so-called Seiberg-Witten map [5, 10, 11]. On a formal level, this can be generalized to spaces with a Moyal-Weyl star product in the context of deformation quantization [12].

While $\mathbb{R}^n_\theta$ appears to be one of the simplest quantum spaces, it has several drawbacks:

- rotation invariance is lost.
- the desired regularization of the UV divergences does not occur. Worse yet, there is a new phenomenon known as UV/IR mixing, which appears to destroy perturbative renormalizability.
- there are serious mathematical complications which are related to the use of deformation quantization.

In view of this, it seems that the apparent simplicity of the quantum space $\mathbb{R}^n_\theta$ is not borne out, and we want to look for other, "healthier" quantum spaces. The first problem to overcome is the lack of symmetry. We therefore insist on quantum spaces with some kind of generalized symmetry; in our context, this will be a symmetry under a quantum group. Next, we insist that no divergences occur at all. This may seem too much to ask for, but it will be satisfied since the spaces under consideration here admit only a finite (but sufficiently large) number of modes. This finiteness property is extremely helpful, because no intuition is available at this stage which could allow to circumvent the difficult technical aspects of infinite-dimensional non-commutative algebras. Nevertheless, one can obtain e.g. $\mathbb{R}^2_\theta$ from these spaces by a scaling procedure, as we will see.

In this thesis, we study a class of quantum spaces which satisfy the above requirements. Moreover, we will argue that these spaces are realized in string theory, in the form of D-branes on (compact) group manifolds G. Indeed, G always carries a non-vanishing B-field for consistency reasons, due the WZW term. Therefore these branes are noncommutative spaces. They have been studied from various points of views [8, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], leading to a nice and coherent picture. On the quasi-classical level they[1] are adjoint orbits in G, for example 2-spheres, or higher-dimensional analogs thereof. The exact description on the world-sheet level is given by a WZW model, which is a 2-dimensional conformal field theory (CFT). This is one of the few situations where string theory on a curved space is well under control. It is known that without D-branes, the WZW model leads to a chiral algebra of left and right currents on G, which satisfy affine Lie algebras $\widehat{\mathfrak{g}}_{L,R}$ at level $k \in \mathbb{N}$. These $\widehat{\mathfrak{g}}_{L,R}$ in turn are closely related to the quantum groups (more precisely quantized universal enveloping algebras) $U_q(\mathfrak{g}_{L,R})$ for $q = exp(\frac{i\pi}{k+g^\vee})$. However, this quantum group structure disappears once the left and right currents are combined into the full currents of *closed* string theory. This changes in the presence of D-branes, which amount to a boundary condition for the *open* strings ending on them, relating the left with the right chiral current. Then only one copy of the affine Lie algebras (the "vector" affine Lie algebra $\widehat{\mathfrak{g}}_V$) survives, which preserves the (untwisted) branes D,

[1]we consider only the simplest type of D-branes here

reflecting the geometric invariance of D under the adjoint action of G. Now the corresponding quantum group $U_q(\mathfrak{g}_V)$ is manifest, and the brane turns out to be a q-deformed space.

We present here a simple and compact description of these quantized D-branes on the classical (compact, simple) matrix groups G of type $SU(N), SO(N)$ and $USp(N)$, and study field theory on them in some simple cases. We first give an algebraic description of the quantized group manifold G, different from the one proposed by Faddeev, Reshetikhin and Takhtajan [24] and Woronowicz [25]. Instead, it is based on the so-called reflection equation [26, 27]. This quantized group manifold turns out to admit an array of discrete quantized adjoint orbits, which are quantum sub-manifolds described by finite operator algebras. Their position in the group manifold and hence their size is quantized in a particular way. This array of quantized adjoint orbits turns out to match precisely the structure of D-branes on group manifolds as found in string theory. Such non-commutative D-branes were first found - in a somewhat different formulation - in the work [8] of Alekseev, Recknagel and Schomerus. We proposed more generally in [28] the quantum-algebraic description of these D-branes. The essential new feature of our approach is the covariance under a quantum analog of the full group of motions $G_L \times G_R$. This allows to address global issues on G, rather than just individual branes.

In the simplest case of the group $SU(2)$, these quantized orbits are q-deformations of the fuzzy spheres S^2_N introduced by John Madore [29], embedded in G. In the present context q is necessarily a root of unity, and is related to the "radius" of the group G: The limit $q \to 1$ corresponds to infinite radius or zero curvature of G, where the D-branes coincide with the "standard" fuzzy spheres. Keeping $q \neq 1$ basically describes the effects of curvature on the group manifold, in a very remarkable way which will be discussed in detail.

This thesis is organized as follows. We start by giving in Chapter 1 the general description of quantized ajoint orbits on quantized group manifolds, including a brief review of the main results from string theory (more precisely conformal field theory). This first chapter is probably the most difficult one; however, it is not indispensible for the remainder of this thesis. We chose this approach because the algebraic description of the quantized group manifolds is strikingly simple and compact, once the mathematical background has been digested. Many characteristic quantities of these quantum spaces will be calculated and compared with results from string theory, with very convincing agreement. Of course, no knowledge of string theory is needed to understand the mathematical constructions of these quantum spaces, and the remaining chapters can be read independently.

In Chapter 2, we proceed to study field theory on the simplest of these space, the q-deformed fuzzy spheres $S^2_{q,N}$, on the first-quantized level. While scalar fields offer little surprise, the formulation of gauge theories naturally leads to Yang-Mills and Chern-Simons type actions. The gauge fields contain an additional scalar degree of freedom, which cannot be disentangled from the "usual" gauge fields. This is matched by the corresponding analysis of gauge theory on fuzzy 2-branes on $SU(2)$ [30], which is possible in a certain limit. The kinetic term of the Yang-Mills action arises dynamically, du to a beautiful mechanism which is similar to spontaneous symmetry breaking. This can also be related to the "covariant coordinates" introduced in [31]. Indeed, the

formulation of gauge theories on this (and other) noncommutative space is in a sense much simpler than the classical description in terms of connections on fiber bundles. This suggests that if properly understood, noncommutative gauge theories should be simpler than commutative ones, rather than more complicated.

There are other aspects of field theory which are intrinsically related to the quantum nature of the space. For example, we find transitions between different geometric phases of gauge theory on S^2_N, which are parallel to instantons in ordinary gauge theories. These are described briefly in Chapter 3.

The next step is to study quantized field theories. We first consider the undeformed fuzzy sphere S^2_N, putting $q = 1$. Chapter 4 is a review of the first one-loop calculation for QFT on S^2_N, which was done in [32] for a scalar φ^4 theory. This was originally motivated by trying to understand the so-called "UV/IR mixing", which is a rather mysterious and troublesome phenomenon in quantum field theories on the quantum plane $\mathbb{R}^n_\theta$. It turns out that $\mathbb{R}^2_\theta$ can be approximated by "blowing up" S^2_N near the north pole, and we are able to understand how the UV/IR mixing arises as the limit of an intriguing "non-commutative anomaly", which is a discontinuity of the quantized (euclidean) field theory as the deformation is sent to zero.

This application nicely illustrates the long-term goal of this line or research, which is to take advantage of the rich mathematical structure of fuzzy D-branes in order to get new handles on quantum field theory. One might of course wonder whether adjoint orbits on Lie groups are of great physical interest. We claim that they could be of considerable interest: There exist 4-dimensional versions such as $\mathbb{C}P^2$, and various limits of them can be taken. Furthermore, one could also consider non-compact groups, and look for branes with Minkowski signature. Knowing that there exist Yang-Mills type gauge theories on these branes [30], this should lead to new ways of formulating physically interesting gauge theories, and quite possibly to new insights into old problems.

Finally, we should point out that quantizations of *co*-adjoint orbits have been considered for a long time [33], [34]. This is *not* what we do here. The quantized adjoint orbits considered here are intimately related to the WZW 3-form on a group manifold, and their description in terms of generalized Poisson-structures is quite involved [35].

This thesis is based on several publications [28, 36, 37, 32, 38] which were written in collaboration with Chong-Sun Chu, Harald Grosse, John Madore, Marco Maceda and Jacek Pawelczyk. I have considerably rewritten and adapted them to make this thesis a coherent work, and added additional material throughout to make it more accessible. Nevertheless, some familiarity with Hopf algebras and quantum groups is assumed. There are many good textbooks available on this subject [39, 40, 41], and a short summary of the main aspects needed here is given in Chapter 1. Chapters 3 and 4 do not involve quantum groups at all.

Acknowledgements

I am indebted to Julius Wess for letting me join his group in Munich allowing me to pursue research which is not always in line with the fashion, and for sharing his deep insights into theoretical physics. Part of the work related to this thesis was done at the Laboratoire de Physique Théorique in Orsay, where I enjoyed being hosted for one year by John Madore, and during several useful visits at the Erwin Schrödinger Institut in Vienna.

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Finally, the ground of this thesis has been laid many years ago in Berkeley, where Bruno Zumino sparked my interest in quantum groups, which has not decreased since then.

Chapter 1

An algebraic description of quantized D-branes on group manifolds

In this chapter[1] we exhibit a certain class of quantum spaces, namely quantized adjoint orbits on (simple, compact) Lie groups G. Some of them will be studied in more detail in the later chapters. We shall not only describe these spaces, but also their embedding in the quantized group manifold, which is an essential part of the mathematical construction. This will allow us to relate them in a convincing way to certain quantized D-branes on G in the framework of string theory. There, D-branes are by definition submanifolds in the target space (which is $G \times M$ for some suitable manifold M), which are expected to be noncommutative spaces.

We shall in fact argue that our quantum spaces are *the* appropriate description of these D-branes. This provides valuable physical insights; in particular, the open strings ending on D-branes are expected to induce a gauge theory on them. One can therefore expect that gauge theories on these quantum spaces exist and are physically meaningful.

The simplest class of D- branes on group manifolds G corresponds to quantized adjoint orbits. We will see that their structure and properties can be described in a compact and simple way in terms of certain q-deformed quantum algebras, which have been considered in various contexts before. "q-deformed" here refers to the fact that they are covariant under the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G, as defined by Drinfeld and Jimbo [42, 43]. Our description will reproduce essentially all known characteristics of stable branes as found in the framework of WZW models on G, in particular their configurations in G, the set of harmonics, and their energies. It covers both generic and degenerate branes.

The quantized D-branes constructed here are hence quantizations of adjoint orbits on G. This might be somewhat puzzling to the reader: It is well-known that there exists a canonical Poisson structure on co-adjoint orbits, which live in the dual $\mathfrak{g}^*$ of the Lie algebra of G; the quantization of these is in fact related to ours. However, this is not what we do, and there is no canonical Poisson structure on adjoint orbits in G. Rather, there is a more complicated so-called "twisted Dirac structure" on G, where the twisting is basically given by the Wess-Zumino

[1]This chapter is based on the joint work [28] with J. Pawelczyk.

3-form on the group. These differential-geometric aspects have been investigated in [35]. Here we decide to circumvent these rather involved classical considerations by starting directly with the full quantum version, which turns out to be much simpler than the semi-classical one. I find this quite gratifying: since the real world is quantum, one should start with the quantum theory and then derive its classical limit, rather than the other way round.

Before discussing the quantum spaces, we first recall some general aspects of noncommutative D-branes in string theory. We then provide some mathematical background on adjoint orbits, and review the main properties of the corresponding D-branes in WZW models in Section 1.3. The account on conformal field theory (CFT) is not self-contained, since it is not required to understand the remainder of this thesis. Rather, these results are used as a guideline, and to connect our algebraic framework with string theory. In order to make this chapter more readable, we will sometimes postpone the technical details to a “local” appendix in Section 1.9 .

1.1 Noncommutative D-branes in string theory

We recall some aspects of noncommutative D-branes in string theory. This is not intended as an introduction to this topic, which has become a very active field of research in recent years.

D-branes in string theory are by definition subvarieties of the target space (which is $T = G \times M$ here for some suitable manifold M) on which open strings can end. The properties of these strings are governed by the action

$$S = \int_\Sigma g^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu} + i\varepsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu} \tag{1-1}$$

Here Σ is the wordsheet of the string which has D as a boundary, $X : \Sigma \to T$ is the embedding of the string in T, $G_{\mu\nu}$ is the metric on the target space T, and $B_{\mu\nu}$ is the crucial antisymmetric 2-form field in T. The structure of such D-branes with a nonvanishing B field background has attracted much attention, because they turn out to be noncommutative spaces [3, 4, 6], at least in a suitable limit. Indeed, the B field is topological in the bulk (if $dB = 0$), and it can be shifted to the boundary of the worldsheet, which is the D-brane. There it induces a Poisson-structure, and the brane becomes a quantization of this Poisson structure. The noncommutative algebra of functions on the brane can be extracted from the algebra of boundary vertex operators. The simplest case of flat branes in a constant B background has been studied extensively (see e.g. [44] for a review), and leads to quantum spaces with a Moyal-Weyl star product corresponding to the constant Poisson structure. This was later generalized to non-constant, closed B [12].

A more complicated situation arises on group manifolds G which carry D-branes, because the B field is not closed any more, but satisfies $dB = H = const \neq 0$ where H is the WZW term. One should therefore expect that the resulting quantization of the branes is more “radical” than just a deformation quantization. Indeed, they turn out to be some generalizations of “fuzzy” spheres with finitely many degrees of freedom. The best approach to study this situation is BCFT, which is an exact description of the worldsheet theory in the presence of the D-brane.

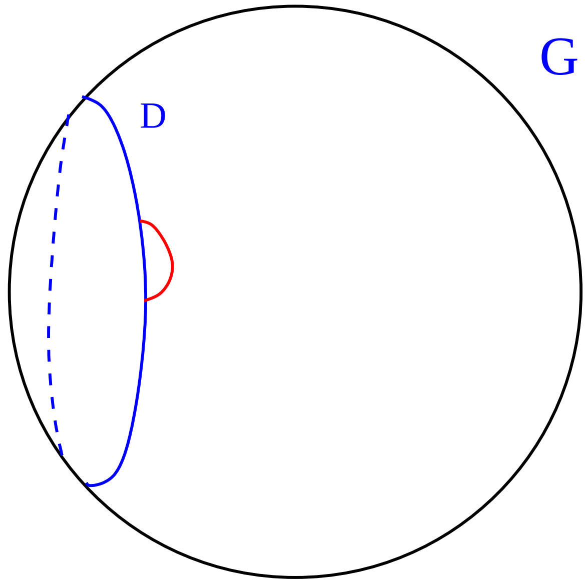

Figure 1.1: String ending on D-brane in G

Other approaches include the Dirac-Born-Infeld (DBI) action, which can be used in the limit of large radius of G. Using these methods, it has been shown in [13, 19, 18, 15] that stable branes can wrap certain conjugacy classes in the group manifold. This will be explained in more detail in Section 1.3. On the other hand, matrix model [9] and again CFT calculations [30] led to an intriguing picture where, in a special limit, the macroscopic branes arise as bound states of $D0$-branes, i.e. zero-dimensional branes.

Attempting to reconcile these various approaches, we proposed in [45] a matrix description of D - branes on $SU(2)$. This led to a "quantum" algebra based on quantum group symmetries, which reproduced all static properties of stable D-branes on $SU(2)$.

In the following chapter, we generalize the approach of [45], and present a simple and convincing description of all (untwisted) D-branes on group manifolds G as noncommutative, i.e. quantized spaces, in terms of certain "quantum" algebras related to $U_q(\mathfrak{g})$. This is analogous to the quantization of flat branes in a constant background B using star products. As in the latter case, our description is based on exact results of boundary conformal field theory (BCFT). We shall show that a simple algebra which was known for more than 10 years as reflection equation (RE) reproduces exactly the same branes as the exact, but much more complicated CFT description. It not only reproduces their configurations in G, i.e. the positions of the corresponding conjugacy classes, but also the (quantized) algebra of functions on the branes is essentially the one given by the CFT description. Moreover, both generic and degenerate branes are predicted, again in agreement with the CFT results. For example, we identify branes on $SU(N+1)$ which are quantizations of $\mathbb{C}P^N$, and in fact q-deformations of the fuzzy $\mathbb{C}P^N$ spaces constructed in [46, 47].

We will not attempt here to recover all known branes on G, such as twisted branes or "type B branes" [48], but concentrate on the untwisted branes. Given the success and simplicity of our description, it seems quite possible, however, that these other branes are described by RE as well.

1.2 The classical geometry of D–branes on group manifolds

1.2.1 Some Lie algebra terminology

We collect here some basic definitions, in order to fix the notation. Let $\mathfrak{g}$ be a (simple, finite-dimensional) Lie algebra, with Cartan matrix $A_{ij} = 2\frac{\alpha_i\cdot\alpha_j}{\alpha_j\cdot\alpha_j}$. Here α_i are the simple roots, and $\cdot$ is the Killing form which is defined for arbitrary weights. The generators $X_i^\pm, H_i$ of $\mathfrak{g}$ satisfy the relations

$$\begin{aligned} [H_i, H_j] &= 0, \quad [H_i, X_j^\pm] = \pm A_{ji} X_j^\pm, & (1\text{-}2)\\ [X_i^+, X_j^-] &= \delta_{i,j} H_i & (1\text{-}3)\end{aligned}$$

The length of a root (or weight) α is defined by $d_\alpha = \frac{\alpha\cdot\alpha}{2}$. We denote the diagram automorphisms by γ, which extend to $\mathfrak{g}$ by $\gamma(H_i) = H_{\gamma(i)},\ \gamma(X_i^\pm) = X_{\gamma(i)}^\pm$. For any root α, the reciprocal root is denoted by $\alpha^\vee = \frac{2\alpha}{(\alpha\cdot\alpha)}$. The dominant integral weights are defined as

$$P_+ = \{\sum n_i\Lambda_i;\ \ n_i \in \mathbb{N}\}, \tag{1-4}$$

where the fundamental weights Λ_i satisfy $\alpha_i^\vee \cdot \Lambda_j = \delta_{ij}$. The Weyl vector is the sum over all positive roots, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. For any weight λ, we define $H_\lambda \in \mathfrak{g}$ to be the Cartan element which takes the value $H_\lambda v_\mu = (\lambda\cdot\mu)\ v_\mu$ on a weight vectors v_μ in some representation. In particular, $H_i = H_{\alpha_i^\vee}$.

For a positive integer k, one defines the "fundamental alcove" in weight space as

$$\boxed{P_k^+ = \{\lambda \in P^+;\ \lambda\cdot\theta \leq k\}} \tag{1-5}$$

where θ is the highest root of $\mathfrak{g}$. It is a finite set of dominant integral weights. For $G = SU(N)$, this is explicitly $P_k^+ = \{\sum n_i\Lambda_i;\ \sum_i n_i \leq k\}$. We shall normalize the Killing form such that $d_\theta = 1$, so that the dual Coxeter number is given by $g^\vee = (\rho + \frac{1}{2}\theta)\cdot\theta$, which is N for $SU(N)$.

We will consider only finite-dimensional representations of $\mathfrak{g}$. V_λ denotes the irreducible highest-weight module of G with highest weight $\lambda \in P_+$, and V_{λ^+} is the conjugate module of V_λ. The defining representation of $\mathfrak{g}$ for the classical matrix groups $SU(N)$, $SO(N)$, and $USp(N)$ will be denoted by V_N, being N-dimensional.

1.2.2 Adjoint orbits on group manifolds

Let G be a compact matrix group of type $SU(N)$, $SO(N)$ or $USp(N)$, and $\mathfrak{g}$ its Lie algebra. For simplicty we shall concentrate on $G = SU(N)$, however all constructions apply to the other cases as well, with small modifications that will be indicated when necessary.

(Twisted) adjoint orbits on G have the form

$$\boxed{\mathcal{C}(t) = \{gt\gamma(g)^{-1}; \quad g \in G\}.} \tag{1-6}$$

Here γ is an auto-morphism of G.In this work we shall study only untwisted branes corresponding to $\gamma = id$, leaving the twisted case for future investigations. One can assume that t belongs to a maximal torus T of G, i.e. that t is a diagonal matrix for $G = SU(N)$. As explained in Section 1.9.2, $\mathcal{C}(t)$ can be viewed as homogeneous space:

$$\mathcal{C}(t) \cong G/K_t. \tag{1-7}$$

Here $K_t = \{g \in G : \ [g,t] = 0\}$ is the stabilizer of $t \in T$. "Regular" conjugacy classes are those with $K_t = T$, and are isomorphic to G/T. In particular, their dimension is $dim(\mathcal{C}(t)) = dim(G) - rank(G)$. "Degenerate" conjugacy classes have a larger stability group K_t, hence their dimension is smaller. Examples of degenerate conjugacy classes are $\mathbb{C}P^N \subset SU(N+1)$, and the extreme case is a point $\mathcal{C}(t=1)$.

Symmetries. The group of motions on G is the product $G_L \times G_R$, which act on G by left resp. right multiplication. It contains the "vector" subgroup $G_V \hookrightarrow G_L \times G_R$, which is diagonally embedded and acts on G via conjugation. In general, the motions rotate the conjugacy classes on the group manifold. However all conjugacy classes are invariant under the adjoint action of G_V,

$$G_V \mathcal{C}(t) G_V^{-1} = \mathcal{C}(t). \tag{1-8}$$

We want to preserve this symmetry pattern in the quantum case, in a suitable sense.

The space of harmonics on $\mathcal{C}(t)$ A lot of information about the spaces $\mathcal{C}(t)$ can be obtained using harmonic analysis, i.e. by decomposing functions on $\mathcal{C}(t)$ into irreps under the action of the (vector) symmetry G_V. This is particularly useful here, because quantized spaces are described in terms of their algebra of functions. The decomposition of this space of functions $\mathcal{F}(\mathcal{C}(t))$ into harmonics can be calculated explicitly using (1-7), and it must be preserved after quantization, at least up to some cutoff. Otherwise, the quantization would not be admissible. One finds (see Section 1.9.2 and [19])

$$\mathcal{F}(\mathcal{C}(t)) \cong \bigoplus_{\lambda \in P_+} mult_{\lambda^+}^{(K_t)} \, V_\lambda. \tag{1-9}$$

Here $mult_{\lambda^+}^{(K_t)} \in \mathbb{N}$ is the dimension of the subspace of V_{λ^+} which is invariant under K_t.

1.2.3 Characterization of the stable D–branes

From the CFT [13, 19] considerations as reviewed in Section 1.3, it follows that there is only a finite set of stable D–branes on G (up to global motions), one for each integral weight $\lambda \in P_k^+$ in the fundamental alcove (1-5). They are given by $\mathcal{C}(t_\lambda)$ for

$$\boxed{t_\lambda = \exp(2\pi i \tfrac{H_\lambda + H_\rho}{k + g^\vee})} \tag{1-10}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector. The restriction to $\lambda \in P_k^+$ follows from the fact that different integral λ may label the same conjugacy class, because the exponential in (1-10) is periodic. This happens precisely if the weights are related by the affine Weyl group, which is generated by the ordinary Weyl group together with translations of the form $\lambda \to \lambda + (k + g^\vee)\frac{2\alpha_i}{(\alpha_i \cdot \alpha_i)}$. Hence one should restrict the weights to be in the fundamental domain of this affine Weyl group, which is precisely the fundamental alcove P_k^+ (1-5). The resulting pattern of stable branes is sketched in Figure 1.2 for the $G = SU(2)$. This case is discussed in more detail in Section 1.7.2.

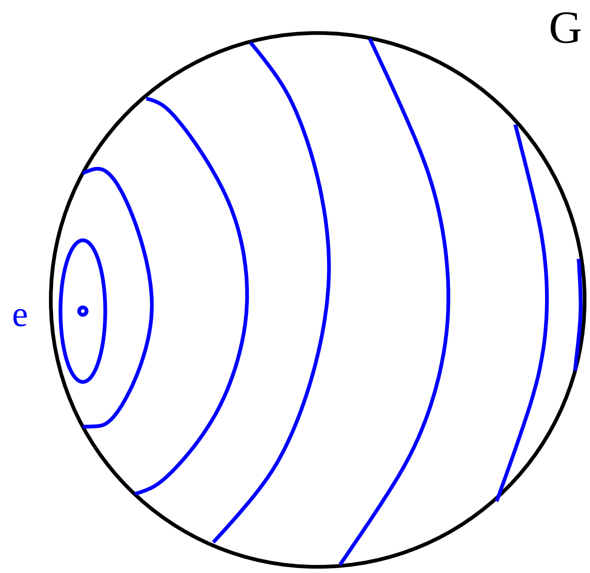

Figure 1.2: D-branes in G

We are interested not only in the branes themselves, but also in their location in G. Information about the location of these (untwisted) branes in G is provided by the quantities

$$s_n = \mathrm{tr}(g^n) = \mathrm{tr}(t^n), \quad g \in \mathcal{C}(t) \tag{1-11}$$

which are invariant under the adjoint action (1-8). The trace is over the defining representation V_N ($= V_{\Lambda_1}$ in the case of $SU(N)$, where Λ_1 is the fundamental weight) of the matrix group G, of dimension N. For the conjucacy classes $\mathcal{C}(t_\lambda)$, they can be calculated easily:

$$\boxed{s_n = \mathrm{tr}_{V_N}\,(q^{2n(H_\rho + H_\lambda)}) = \sum_{\nu\in V_N} q^{2n(\rho+\lambda)\cdot\nu}} \tag{1-12}$$

where

$$q = e^{\frac{i\pi}{k+g^\vee}}.$$

The s_n are independent functions of the weight λ for $n = 1, 2, ..., rank(G)$, which completely characterize the class $\mathcal{C}(t_\lambda)$. These functions have the great merit that their quantum analogs (1-63) can also be calculated explicitly.

An equivalent characterization of these conjugacy classes is provided by a characteristic equation: for any $g \in \mathcal{C}(t_\lambda)$, the relation $P_\lambda(g) = 0$ holds in $Mat(V_N, \mathbb{C})$, where P_λ is the polynomial

$$P_\lambda(x) = \prod_{\nu \in V_N} (x - q^{2(\lambda+\rho)\cdot\nu}). \tag{1-13}$$

This follows immediately from (1-10), since t_λ has the eigenvalues $q^{2(\lambda+\rho)\cdot\nu}$ on the weights ν of the defining representation V_N. Again, we will find analogous characteristic equations in the quantum case.

We should perhaps point out that co-adjoint orbits on Lie groups carry a natural symplectic structure, and the quantization of these spaces has been considered before [33], [34]. As individual manifolds, the quantized D-branes we shall find are related to these quantized (co)adjoint orbits, which by itself would perhaps not be too exciting. The point is, however, that our description not only gives isolated quantized orbits, but a quantization of the entire group manifold, and the quantized orbits are embedded at precisely the positions as predicted by string theory. This means that our construction will *global* and reproduces the orbits correctly not only near the origin, but also reproduces correctly the shrinking of the branes beyond the "equator" of G, as they approach the point-liks brane at $-id \in G$. This works only for q a root of unity, and is the major result of our construction.

1.3 The CFT description of D-branes in the WZW model

This section reviews some results from the CFT description of branes in WZW models on G, and is *not* self-contained. It serves as inspiration to the algebraic considerations below, and to establish the relations with string theory. All results presented here are well known. They are not necessary to understand the rest of this chapter, and the reader who is not familiar with CFT and string theory may skip this section and go directly to the Section 1.5

The WZW model (see e.g. [49, 50]) is a CFT with a Lie group G as target space, and action (1-1)

$$S = \int_\Sigma g^{-1}dgg^{-1}dg + B, \qquad dB = k(g^{-1}dg)^3. \tag{1-14}$$

It is specified by a level k of the affine Lie algebra $\widehat{\mathfrak{g}}$ whose horizontal subalgebra is $\mathfrak{g}$, the Lie algebra of G. We shall consider only simple, compact groups G, so that the level k must be a positive integer. On an algebraic level, the WZW branes can be described by boundary states $|B\rangle\rangle \in \mathcal{H}^{\text{closed}}$ respecting a set of boundary conditions. A large class of boundary conditions is of the form

$$\Big(J_n + \gamma(\tilde{J})_{-n}\Big)|B\rangle\rangle = 0 \qquad n \in \mathbf{Z} \tag{1-15}$$

where γ is an auto-morphism of the affine Lie algebra $\widehat{\mathfrak{g}}$. Here J_n are the modes of the left-moving currents, and $\tilde{J}_n$ are the modes of the right-moving currents. Boundary states with $\gamma = 1$ are called "symmetry-preserving branes" or "untwisted branes": these are the object of interest

here. The untwisted ($\gamma = 1$) boundary condition (1-15) breaks half of the symmetries $\widehat{\mathfrak{g}}_L \times \widehat{\mathfrak{g}}_R$ of the WZW model down to the vector part $\widehat{\mathfrak{g}}_V$. This will be important for the choice of the relevant quantum symmetry $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$ (or $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$) in the algebraic considerations of Section 1.5.

The condition (1-15) alone does not define a good boundary state: one also has to impose open-closed string duality of the amplitude describing interactions of branes. This leads to so called Cardy (boundary) states. For the untwisted case they are labeled by $\lambda \in P_k^+$ corresponding to integrable irreps of $\widehat{\mathfrak{g}}$, which are precisely the weights in the "fundamental alcove" (1-5). Therefore the stable branes in the WZW model are in one-to-one correspondence to the weights $\lambda \in P_k^+$.

The CFT description yields furthermore an important formula for the energy of the brane corresponding to λ:

$$\boxed{E_\lambda = \prod_{\alpha>0} \frac{\sin\left(\pi \frac{\alpha\cdot(\lambda+\rho)}{k+g^\vee}\right)}{\sin\left(\pi \frac{\alpha\cdot\rho}{k+g^\vee}\right)}.} \tag{1-16}$$

For $k \gg N$, one can expand the denominator in (1-16) to obtain a formula which compared with the DBI description [20] shows that the leading k-dependence fits perfectly with the interpretation of a brane wrapping once a conjugacy class $\mathcal{C}(t_\lambda)$ given by the element t_λ (1-10) of the maximal torus of G.

The BCFT is also the basis for the description of branes as noncommutative spaces, in complete analogy to the case of flat branes in a constant B field as explained in Section 1.1. The properties of the D-branes are determined by open strings ending on them, which are described by boundary vertex operators. The relevant operators here are the primary fields of the BCFT which transform under the unbroken symmetry algebra $\widehat{\mathfrak{g}}_V \subset \widehat{\mathfrak{g}}_L \times \widehat{\mathfrak{g}}_R$. Their number is finite for any compact WZW model. They satisfy an operator-product algebra (OPE), which has the form

$$Y_i^I(x)\, Y_j^J(x') \sim \sum_{K,k} (x-x')^{h_I+h_J-h_K} \begin{bmatrix} I & J & K \\ i & j & k \end{bmatrix} \begin{Bmatrix} I & J & K \\ \lambda & \lambda & \lambda \end{Bmatrix}_q Y_k^K(x') + \ldots \tag{1-17}$$

where we assume $G = SU(2)$ for simplicity[2]. Here $h_I = I(I+1)/(k+g^\vee)$ are the conformal dimension of the primaries Y_i^I, and the sum over K has a cutoff $K_{\max} = \min(I+J, k-I-J, 2\lambda, k-2\lambda)$. It involves the $3j$ symbols $[...]$ of $su(2)$, and the $6j$ symbols $\{...\}_q$ of $U_q(su(2))$, the quantized universal enveloping algebra of $su(2)$. The variables x, x' are on the boundary of the world sheet Σ, which is usually taken to be the upper half plane. The dots denote operators (descendants) which do not enter the description of the brane as a manifold. For higher groups, the OPE has a similar form [19] involving the corresponding $3j$ and $6j$ symbols of $\mathfrak{g}$ and $U_q(\mathfrak{g})$, respectively. This indicates the deep relation between affine Lie algebras and quantum groups at roots of unity [51, 39, 52].

For $k \to \infty$, the conformal weights h_I become zero, and this OPE reduces to the algebra of functions on the fuzzy sphere S_N^2 in the case $G = SU(2)$ (see also Chapter 4). The Y_i^I

[2]We use the $su(2)$ convention here that the weight λ corresponds to half-integers. This should not cause any confusion, because the $6j$ symbols depend on representations rather than numbers.

then play the role of the spherical harmonics. This interpretation was extended in [19] to branes on arbitrary G. For finite k, this interpretation is less clear, because the conformal dimensions are nonvanishing, and the truncation of the OPE may seem unjustified. We shall nevertheless consider the brane with label λ as a quantized manifold, whose space of function is encoded in the finite set $\{Y_i^I\}$ of primaries.

To extract an algebra of functions on the branes, we note that the variables x, x' in the above OPE are only world-sheet variables and not physical. It was therefore suggested in [8] in the case $G = SU(2)$ to associate with this OPE the "effective" algebra

$$Y_i^I Y_j^J = \sum\nolimits_{K,k} \left[\begin{array}{ccc} I & J & K \\ i & j & k \end{array} \right] \left\{ \begin{array}{ccc} I & J & K \\ \lambda & \lambda & \lambda \end{array} \right\}_q Y_k^K, \tag{1-18}$$

interpreted as algebra of functions on the brane, which is covariant under $su(2)$. Unfortunately this algebra is only quasi-associative, due to the curious mixing of undeformed and q-deformed group theoretical objects. However, as explained in Sections 1.4.5 and 1.6.5, it is equivalent by a Drinfeld-twist to the associative algebra

$$\boxed{Y_i^I \star Y_j^J = \sum\nolimits_{K,k} \left[\begin{array}{ccc} I & J & K \\ i & j & k \end{array} \right]_q \left\{ \begin{array}{ccc} I & J & K \\ \lambda & \lambda & \lambda \end{array} \right\}_q Y_k^K,} \tag{1-19}$$

which is covariant under $U_q(su(2))$. Associativity of course makes this twisted algebra much easier to work with, and appears to be the most natural description. The twisting of the algebra could presumably be realized on the CFT level by introducing some kind of "dressed" vertex operators, similar as in [53]. For other groups G, the relevant algebras are entirely analogous, involving the generalized $3j$ and $6j$ symbols of $U_q(\mathfrak{g})$.

We will recover exactly the same algebra (1-19) from the quantum-algebraic approach below, based on symmetry principles involving quantum groups. This is the basis of our algebraic description in the following sections. It was indeed known for a long time [51] that the $6j$ symbols of $U_q(\mathfrak{g})$ enter the chiral OPE's of WZW models, but they become "invisible" in the full CFT. The remarkable thing here is that on the D-branes, the q-deformation becomes manifest, because the chiral symmetry $\widehat{\mathfrak{g}}_L \times \widehat{\mathfrak{g}}_R$ is broken down to $\widehat{\mathfrak{g}}_V$. Before proceeding, we must therefore pause and collect some relevant facts about $U_q(\mathfrak{g})$ and related algebraic structures.

1.4 Some quantum group technology

We recall here the basic definitions and properties of quantum groups which are needed below. Some the topics will be elaborated further in later chapters. This section should also serve as some kind of "background check" for the reader: while it might help as a first crash course to get some basic understanding, the presentation is definitely not self-contained. For a more detailed account, the reader is referred to some of the existing monographs, for example [41, 39, 49].

1.4.1 $U_q(\mathfrak{g})$

Given a (simple, finite-dimensional) Lie algebra $\mathfrak{g}$ as in Section 1.2.1, the quantized universal enveloping algebra[3] [43, 42] $U_q(\mathfrak{g})$ is generated by the generators $X_i^{\pm}, H_i$ satisfying the relations

$$
\begin{aligned}
[H_i, H_j] &= 0, \quad [H_i, X_j^{\pm}] = \pm A_{ji} X_j^{\pm}, && \text{(1-20)}\\
[X_i^+, X_j^-] &= \delta_{i,j} \frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} && \text{(1-21)}
\end{aligned}
$$

plus analogs of the Serre relations. Here $d_i = \frac{\alpha_i \cdot \alpha_i}{2}$ is the length of the root α_i. The Hopf algebra structure of $U_q(\mathfrak{g})$ is given by the comultiplication and antipode

$$
\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \qquad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{d_i H_i/2} + q^{-d_i H_i/2} \otimes X_i^{\pm},\\
S(H_i) &= -H_i, \qquad S(X_i^{\pm}) = -q^{\pm d_i} X_i^{\pm}. && \text{(1-22)}
\end{aligned}
$$

The coproduct is conveniently written in Sweedler-notation as $\Delta(u) = u_1 \otimes u_2$, for $u \in U_q(\mathfrak{g})$, where a summation is implied. Generators $X_\alpha^{\pm}$ corresponding to the other roots can be defined as well. It is easy to verify the important relation $S^2(u) = q^{2H_\rho}\ u\ q^{-2H_\rho}$ which holds for all $u \in U_q(\mathfrak{g})$. The *quasitriangular structure* of $U_q(\mathfrak{g})$ is given by the universal $\mathcal{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, which satisfies the properties

$$
\begin{aligned}
(\Delta \otimes id)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \quad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} && \text{(1-23)}\\
\Delta'(x) &= \mathcal{R}\Delta(x)\mathcal{R}^{-1} && \text{(1-24)}
\end{aligned}
$$

where Δ' denotes the reversed coproduct. This implies the Yang-Baxter equation

$$
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1-25}
$$

Introducing the matrices

$$
\begin{aligned}
L^+ &= (id \otimes \pi)(\mathcal{R}),\\
SL^- &= (\pi \otimes id)(\mathcal{R}) && \text{(1-26)}
\end{aligned}
$$

where π is a representation of $U_q(\mathfrak{g})$, the relations (1-21) can be written in the form

$$
\begin{aligned}
R_{12} L_2^+ L_1^+ &= L_1^+ L_2^+ R_{12},\\
R_{12} L_2^- L_1^- &= L_1^- L_2^- R_{12},\\
R_{12} L_2^+ L_1^- &= L_1^- L_2^+ R_{12} && \text{(1-27)}
\end{aligned}
$$

which follow easily from (1-25). Explicitly, $\mathcal{R}$ has the form

$$
\mathcal{R} = q^{H_i (B^{-1})_{ij} \otimes H_j}\ (1 \otimes 1 + \sum U^+ \otimes U^-) \tag{1-28}
$$

[3] we will ignore subtle mathematical issues of topological nature, being interested only in finite-dimensional representations where all series terminate and convergence issues are trivial.

where B is the symmetric matrix $d_j^{-1}A_{ij}$, and U^+, U^- stands for terms in the Borel sub-algebras of rising respectively lowering operators. Therefore L^+ are lower triangular matrices with X_α^+'s below the diagonal, and L^- are upper triangular matrices with X_α^-'s above the diagonal. For $\mathfrak{g} = sl(2)$ one has explicitly

$$L^+ = \begin{pmatrix} q^{H/2} & 0 \\ q^{-\frac{1}{2}}\lambda X^+ & q^{-H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-H/2} & -q^{\frac{1}{2}}\lambda X^- \\ 0 & q^{H/2} \end{pmatrix} \tag{1-29}$$

The above definitions are somewhat sloppy mathematically, q being unspecified. This is justified here because we are only interested in certain finite-dimensional representations of $U_q(\mathfrak{g})$, and we will in fact only consider the case where q is a root of unity,

$$\boxed{q = e^{\frac{i\pi}{k+g^\vee}}.}$$

Hence we are dealing with (some version of the) "finite quantum group" $u_q^{fin}(\mathfrak{g})$ at roots of unity [39].

1.4.2 Representations of $U_q(\mathfrak{g})$

For generic (or formal) q, the representation theory of $U_q(\mathfrak{g})$ is completely parallel to that of $U(\mathfrak{g})$. The situation is very different at roots of unity, many non-classical types of representations arise. Here we consider only the simplest ones, which are the irreducible highest weight representations V_λ with highest weight $\lambda \in P_k^+$ in the fundamental alcove (1-5). We shall call them *regular* representations.

A useful concept at roots of unity is the *quantum dimension* of a representation V,

$$\dim_q(V) = \mathrm{tr}_V(q^{2H_\rho}) \tag{1-30}$$

which for highest-weight representations V_λ with $\lambda \in P_k^+$ can be calculated using Weyls character formula

$$\dim_q(V) = \prod_{\alpha>0} \frac{\sin(\pi\frac{\alpha\cdot(\lambda+\rho))}{k+g^\vee})}{\sin(\pi\frac{\alpha\cdot\rho}{k+g^\vee})}. \tag{1-31}$$

Clearly $\dim_q(V_\lambda) > 0$ for $\lambda \in P_k^+$, i.e. for regular representations. These representations are in one-to-one correspondence with the integrable modules of the affine Lie algebra $\widehat{\mathfrak{g}}$ at level k, see e.g. [49]. This is part of a deep relation between affine Lie algebras and quantum groups at roots of unity [39, 52]. This relation extends to tensor products as follows: In general, the tensor product $V_\lambda \otimes V_{\lambda'}$ is in general not completely reducible. Rather, it decomposes as $V_\lambda \otimes V_{\lambda'} = (\oplus_\mu V_\mu) \oplus T$ where V_μ is regular, and T denotes indecomposable "tilting modules" [39]. It turns out that $\dim_q(T) = 0$, and one can define a "truncated tensor product" $\overline{\otimes}$ by simply omitting all the modules T in the tensor product which have vanishing quantum dimension. It turns out that $\overline{\otimes}$ is associative, and coincides precisely with the fusion rules of integrable modules of $\widehat{\mathfrak{g}}$ at level k. This gives exactly the modes which occur in the rhs of (1-18).

Furthermore, one can show that they are unitary representations with respect to the star-structure

$$H_i^* = H_i, \quad (X_i^{\pm})^* = X_i^{\mp}, \tag{1-32}$$

see [54] for a proof. For $\lambda \notin P_k^+$, this is no longer the case in general, and the corresponding highest weight representations become non-classical.

Invariant tensors Consider some irreducible representation $\pi^a_b(u)$ of $U_q(\mathfrak{g})$. Then g^{ab} is an invariant 2- tensor if

$$\pi(u_1)^a_{a'}\pi(u_2)^b_{b'}g^{a'b'} = g^{ab}\ \varepsilon(u), \tag{1-33}$$

and similarly for higher-order tensors. There exist as many invariant tensors for a given representation as in the undeformed case, because they are in one-to-one correspondence with trivial components in the tensor product of the representation, whose decomposition is the same as classically for generic q. For roots of unity, of course, there may be additional ones. Invariant 2-tensors can be used as usual to rise and lower indices, however one has to be careful with the ordering of indices. For example, if f^{abc} is an invariant 3-tensor, one can define

$$f^{ab}{}_c = f^{abc'}g_{c'c},\ f^a{}_{bc} = f^{ac'b'}g_{b'b}g_{c'c},\ f_{abc} = f^{c'b'a'}g_{a'a}g_{b'b}g_{c'c}. \tag{1-34}$$

The invariance of the tensor with lower indices is then as follows:

$$f_{a'b'c'}\ \pi(Su_3)^{a'}_a\pi(Su_2)^{b'}_b\pi(Su_1)^{c'}_c = f_{abc}\ \varepsilon(u). \tag{1-35}$$

1.4.3 Dual Hopf algebras and $Fun_q(G)$

Two Hopf algebras $\mathcal{U}$ and $\mathcal{G}$ are *dually paired* if there exists a non-degenerate pairing $<\ ,\ >$: $\mathcal{G}\otimes\mathcal{U}\to\mathbb{C}$, such that

$$\begin{aligned} <ab,u> &= <a\otimes b,\Delta(u)>, \qquad <a,uv>=<\Delta(a),u\otimes v>, \\ <S(a),u> &= <a,S(u)>, \quad <a,1>=\varepsilon(a), \quad <1,u>=\varepsilon(u), \end{aligned} \tag{1-36}$$

for $a,b\in\mathcal{G}$ and $u,v\in\mathcal{U}$. The dual Hopf algebra of $U_q(\mathfrak{g})$ is $Fun_q(G)$, as defined in [24]. It is generated by the elements of a $N\times N$ matrix $A=(A^i{}_j)\in Mat(N,Fun_q(G))$, which can be interpreted as quantized coordinate functions on G. The dual evaluation is fixed by $\pi^i{}_j(u) =< A^i{}_j, u >$, where π is the defining representation of $U_q(\mathfrak{g})$. The coalgebra structure on $Fun_q(G_q)$ comes out as classically:

$$\begin{aligned} \Delta A &= A\dot{\otimes}A, \quad \text{i.e.} \quad \Delta(A^i{}_j)=A^i{}_k\otimes A^k{}_j. \\ S(A) &= A^{-1}, \quad \epsilon(A^i_j)=\delta^i_j, \quad \epsilon(A^i_j)=\delta^i_j. \end{aligned} \tag{1-37}$$

The inverse matrices A^{-1} are well-defined after suitable further constraints on A are imposed, as in [24]: for example, there can be an invariant tensor $(\varepsilon_q)_{i_1\dots i_N}$ of $U_q(\mathfrak{g})$ which defines the quantum determinant of $Fun_q(G_q)$,

$$\det{}_q(A)=1 \tag{1-38}$$

where

$$\det_q(A)\ (\varepsilon_q)_{j_1\dots j_N} = (\varepsilon_q)_{i_1\dots i_N}\ A^{i_1}_{j_1}\dots A^{i_N}_{j_N} \tag{1-39}$$

and other constraints for the orthogonal and symplectic cases, which can be found in the literature. One also finds that the matrix elements of A satisfy the commutation relations

$$R^{ij}{}_{kl}A^k{}_mA^l{}_n = A^j{}_sA^i{}_rR^{rs}{}_{mn}, \tag{1-40}$$

which can be written more compactly in tensor product notation as

$$R_{12}A_1A_2 = A_2A_1R_{12}. \tag{1-41}$$

1.4.4 Covariant quantum algebras

Assume that $\mathcal{G}, \mathcal{U}$ are two Hopf algebras. Then an algebra $\mathcal{M}$ is called a *(left) $\mathcal{U}$-module algebra* if there is an action $\triangleright$ of $\mathcal{U}$ on $\mathcal{M}$ such that

$$u \triangleright (xy) = (u \triangleright x) \triangleright y, \quad u \triangleright 1 = \varepsilon(u)1 \tag{1-42}$$

for $u \in \mathcal{U}$. Similarly, $\mathcal{M}$ is a *(right) $\mathcal{G}$-comodule algebra* if there is a coaction $\nabla : \mathcal{M} \to \mathcal{M} \otimes \mathcal{G}$ of $\mathcal{G}$ on $\mathcal{M}$ such that

$$(id \otimes \Delta)\nabla = (\nabla \otimes id)\nabla, \quad (\varepsilon \otimes id)\nabla = id. \tag{1-43}$$

It is easy to see that if $\mathcal{G}, \mathcal{U}$ are dually paired Hopf algebras, then a (right) $\mathcal{G}$ - comodule algebra $\mathcal{M}$ is automatically a (left) $\mathcal{U}$ – module algebra by

$$u \triangleright M = \langle \nabla(M), u\rangle \tag{1-44}$$

where $\langle m \otimes a, u\rangle = m\langle a, u\rangle$, and vice versa.

1.4.5 Drinfeld twists

A given Hopf algebra $\mathcal{U}$ can be *twisted* [55, 56] using an invertible element

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2\ \in\ \mathcal{U} \otimes \mathcal{U}$$

(in a Sweedler notation, where a sum is implicitly understood) which satisfies

$$(\varepsilon \otimes id)\mathcal{F} = \mathbf{1} = (id \otimes \varepsilon)\mathcal{F}, \tag{1-45}$$

This works as follows: Let $\mathcal{U}_\mathcal{F}$ be the Hopf algebra which coincides with $\mathcal{U}$ as algebra, but has the twisted coalgebra structure

$$\Delta_\mathcal{F}(u) = \mathcal{F}\Delta(u)\mathcal{F}^{-1}, \tag{1-46}$$

$$S_\mathcal{F}(u) = \gamma^{-1}S(u)\gamma \tag{1-47}$$

where $\gamma = S(\mathcal{F}_1^{-1})\mathcal{F}_2^{-1}$. $\mathcal{F}$ is said to be a cocycle if the coassociator

$$\phi := [(\Delta \otimes \mathrm{id})\mathcal{F}^{-1}](\mathcal{F}^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)\mathcal{F}] \qquad \in \mathcal{U}^{\otimes 3} \tag{1-48}$$

is equal to the unit element. In that case, one can show that $\Delta_{\mathcal{F}}$ is coassociative, and moreover if $\mathcal{U}$ is quasitriangular, then so is $\mathcal{U}_{\mathcal{F}}$ with

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}. \tag{1-49}$$

The same twist $\mathcal{F}$ can also be used to twist the corresponding (co)module algebras: Assume that $\mathcal{M}$ is a left $\mathcal{U}$-module algebra. Then

$$a \star b := (\mathcal{F}_1^{-1} \triangleright a)\ (\mathcal{F}_2^{-1} \triangleright b) \tag{1-50}$$

defines a new product on the same space $\mathcal{M}$, which turns it into a left $\mathcal{U}_{\mathcal{F}}$-module algebra; see Section (5.3) for more details. Associativity of $\star$ is a consequence of the cocycle condition on $\mathcal{F}$, provided $\mathcal{M}$ is associative.

There is also a more general notion of twisting where $\mathcal{F}$ is not a cocycle, which relates $U_q(\mathfrak{g})$ to $U(\mathfrak{g})$ (or more precisely a related quasi–Hopf algebra. The relation between $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ will be discussed in more detail in Section 5.2). A fundamental theorem by Drinfeld (Proposition 3.16 in Ref. [55]) states that there exists an algebra isomorphism

$$\varphi : U_q(\mathfrak{g}) \to U(\mathfrak{g})[[h]] \tag{1-51}$$

(considering $q = e^h$ as formal for the moment), and a twist $\mathcal{F}$ which relates the undeformed coproduct on $U(\mathfrak{g})$ to the q-deformed one by the formula (1-46). This twist $\mathcal{F}$ (which is not a cocycle) can now be used to twist the quasiassociative $U(\mathfrak{g})$ -module algebra (1-18), and turns it into the associative $U_q(\mathfrak{g})$ -module algebra (1-19). Indeed, from (1-46) it follows that $\mathcal{F}$ relates the undeformed Clebsch–Gordan coefficients to the q-deformed ones:

$$\begin{bmatrix} I & J & K \\ i & j & k' \end{bmatrix} (g^{(K)})^{k'k} = \begin{bmatrix} I & J & K \\ i' & j' & k' \end{bmatrix}_q (g_q^{(K)})^{k'k}\ \ \pi_i^{i'}(\mathcal{F}^{(1)})\pi_j^{j'}(\mathcal{F}^{(2)}). \tag{1-52}$$

Here we have raised indices using $(g_q^{(K)})^{k'k}$, which is the q–deformed invariant tensor. Both $(g_q^{(K)})^{k'k}$ and its underformed counterpart $(g^{(K)})^{k'k}$ will sometimes be suppressed, i.e. absorbed in the Clebsches. It should also be noted that even though the abstract element $\mathcal{F}$ exists only for formal q, the representations of $\mathcal{F}$ on the *truncated* tensor product of "regular" representations V_λ does exist at roots of unity, because the decomposition into irreps as well as the Clebsches are then analytic in q. Hence the twisted multiplication rule (1-50) for the generators Y_i^I is precisely (1-19), which defines an associative algebra[4]. Associativity can be verified either using the pentagon identity for the q-deformed $6j$ symbols similar as in Chapter 4, or simply by an explicit construction of this algebra in terms of operators on an irreducible representation space of $U_q(\mathfrak{g})$. This is explained in Section 1.6.5.

[4]there is a subtle point here: the twist $\mathcal{F}$ is only defined up to "gauge invariance" (5-14), which amounts to the ambiguity of the Clebsch-Gordan coefficients up to a phase. One can fix this ambiguity by requiring the "standard" normalization, which then yields an associative algebra.

1.5 Quantum algebras and symmetries for D-branes

We return to the problem of describing the quantized D-branes on G. To find the algebra of quantized funcitons on G and its D-branes, we shall make an "educated guess", and then verify the desired properties.

The quantum spaces under consideration should certainly admit some appropriate quantum versions of the symmetries G_L, G_R and G_V. In technical terms, this suggests that they should be module algebras $\mathcal{M}$ under some quantum groups. In view of the results in Section 1.3, we expect that these symmetry algebras are essentially $U_q(\mathfrak{g})$ [39], whose representations are known to be parallel to those of $\widehat{\mathfrak{g}}$ of the WZW model [52].

Since we are considering matrix groups G, we further assume that the appropriate (module) algebra $\mathcal{M}$ is generated by matrix elements M^i_j with indices i, j in the defining representation V_N of G, subject to some commutation relations and constraints. With hindsight, we claim that these relations are given by the so-called *reflection equation* (RE) [26, 27], which in short-hand notation reads

$$\boxed{R_{21}M_1R_{12}M_2 = M_2R_{21}M_1R_{12}.} \tag{1-53}$$

Here R is the $\mathcal{R}$ matrix of $U_q(\mathfrak{g})$ in the defining representation. Displaying the indices explicitly, this means

$$(\mathrm{RE})\,{}^{i\,k}_{j\,l}: \qquad R^k{}_a{}^i{}_b\; M^b_c\; R^c{}_j{}^a{}_d\; M^d_l = M^k_a\; R^a{}_b{}^i{}_c\; M^c_d\; R^d{}_j{}^b{}_l\,. \tag{1-54}$$

The indices $\{i, j\}$, $\{k, l\}$ correspond to the first (1) and the second (2) vectors space in (1-53). Because $\mathcal{M}$ should describe a quantized group manifold G, we need to impose constraints which ensure that the branes are indeed embedded in such a quantum group manifold. In the case $G = SU(N)$, these are $\det_q(M) = 1$ where $\det_q$ is the so-called quantum determinant (1-66), and suitable reality conditions imposed on the generators M^i_j. Both will be discussed below.

One can also think of (1-53) as being analogs of the boundary condition (1-15). As we shall see, RE has indeed similar symmetry properties. This is the subject of the the following subsection. We should mention here that RE appeared more then 10 years ago in the context of the boundary integrable models, and is sometimes called boundary YBE [26].

1.5.1 Quantum symmetries of RE

There are 2 equivalent ways to look at the symmetry of $\mathcal{M}$, either as a (right) comodule algebra or as a (left) module algebra under a suitable quantum group. We recall the concepts of Section 1.4.4 here. The symmetry algebras of $\mathcal{M}$ are $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$ and $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$, which are dual Hopf algebras.

1) $\mathcal{M}$ as comodule algebra. The easiest way to find the full symmetry is to postulate that the matrix M transforms as

$$\boxed{M^i_j \to (s^{-1}Mt)^i_j} \tag{1-55}$$

where s^i_j and t^i_j generate the algebras $\mathcal{G}_L$ and $\mathcal{G}_R$ respectively, which both coincide with the well–known quantum groups [24] $Fun_q(G)$ as described in Section 1.4.3. For example, $s_2 s_1 R = R s_1 s_2,\ t_2 t_1 R = R t_1 t_2$.[5] In (1-55) matrix multiplication is understood. It is easy to see that this is a symmetry of the RE if we impose that (the matrix elements of) s and t commute with M, and additionally satisfy $s_2 t_1 R = R t_1 s_2$. Notice that (1-55) is a quantum analog of the action of the classical isometry group $G_L \times G_R$ on classical group element g as in Section (1.2.2).

Symmetries become powerful only because they have a group-like structure, i.e. they can be iterated. In mathematical language this means that we have a Hopf algebra denoted by $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$:

$$s_2 s_1\, R = R\, s_1 s_2, \quad t_2 t_1\, R = R\, t_1 t_2, \quad s_2 t_1\, R = R\, t_1 s_2 \tag{1-56}$$
$$\Delta s = s \otimes s, \quad \Delta t = t \otimes t, \tag{1-57}$$
$$S(s) = s^{-1}, \quad \epsilon(s^i_j) = \delta^i_j, \quad S(t) = t^{-1}, \quad \epsilon(t^i_j) = \delta^i_j \tag{1-58}$$

(here S is the antipode, and ϵ the counit). The inverse matrices s^{-1} and t^{-1} are defined after suitable further (determinant-like) constraints on s and t are imposed, as explained in Section 1.4.3, [24]. Formally, $\mathcal{M}$ is then a right $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$ - comodule algebra; see Section 1.4.4 for further details.

Furthermore, $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$ can be mapped to a vector Hopf algebra $\mathcal{G}_V$ with generators r, by $s^i_j \otimes 1 \to r^i_j$ and $1 \otimes t^i_j \to r^i_j$ (thus basically identifying $s = t = r$ on the rhs). The (co)action of $\mathcal{G}_V$ on the M's is then

$$M^i_j \to (r^{-1} M r)^i_j. \tag{1-59}$$

2) $\mathcal{M}$ as module algebra. Equivalently, we can consider $\mathcal{M}$ as a left module algebra under $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$, which is dual Hopf algebra to $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$. As an algebra, it is the usual tensor product $U_q(\mathfrak{g}_L) \otimes U_q(\mathfrak{g}_R)$, generated by 2 copies of $U_q(\mathfrak{g})$. The dual evaluation $\langle , \rangle$ between $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$ and $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$ is defined componentwise, using the standard dualities of $\mathcal{G}_{L,R}$ with $U_q(\mathfrak{g}_{L,R})$. The Hopf structure turns out to be

$$\begin{aligned} \Delta_{\mathcal{R}} : U_q^L \otimes U_q^R &\;\to\; (U_q^L \otimes U_q^R) \otimes (U_q^L \otimes U_q^R), \\ u^L \otimes u^R &\;\mapsto\; \mathcal{F}(u_1^L \otimes u_1^R) \otimes (u_2^L \otimes u_2^R)\mathcal{F}^{-1} \end{aligned} \tag{1-60}$$

with $\mathcal{F} = 1 \otimes \mathcal{R}^{-1} \otimes 1$. This is a special case of a Drinfeld twist as discussed in Section 1.4.5. The action of $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$ on $\mathcal{M}$ which is dual to (1-55) comes out as

$$(u_l \otimes u_R) \triangleright M^i_j = \pi^i_l(S u_L) M^l_k \pi^k_j(u_R), \tag{1-61}$$

where $\pi()$ is the defining representation V_N of $U_q(\mathfrak{g})$. This is a symmetry of $\mathcal{M}$ in the usual sense, because the rhs is again an element in $\mathcal{M}$. Moreover, there is a Hopf-algebra map $u \in U_q(\mathfrak{g}_V) \to \Delta(u) \in U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$ where Δ is the usual coproduct. This defines the vector sub-algebra $U_q(\mathfrak{g}_V)$. It induces on $\mathcal{M}$ the action

$$u \triangleright M^i_j = \pi^i_k(S u_1) M^k_l \pi^l_j(u_2), \tag{1-62}$$

[5] R with suppressed indices means R_{12}.

which is again dual to the coaction (1-59). At roots of unity, these dualities are somewhat subtle. We will not worry about this, because covariance of the reflection equation under $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$ can also be verified directly.

The crucial points in our construction is the existence of a vector sub-algebra $U_q(\mathfrak{g}_V)$ of $U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_{\mathcal{R}}$ (or the analogous notion for the dual $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$). We will see that the central terms of $\mathcal{M}$ which characterize its representations is invariant only with respect to that $U_q(\mathfrak{g}_V)$ (or $\mathcal{G}_V$). This will allow to interpret these sub-algebras as isometries of the quantum D-branes. In other words, the RE imposes very similar conditions on the symmetries and their breaking as the original BCFT WZW model described in section 1.3 does. This is not the case for other conceivable quantized algebras of funcitons on G, such as $Fun_q(G)$.

1.5.2 Central elements of RE

Next we discuss some general properties of the algebra defined by (1-53). We need to find the central elements, which are expected to characterize its irreps. This problem was solved in the second paper of [57, 27, 58]. The (generic) central elements of the algebra (1-53) are

$$c_n = \mathrm{tr}_q(M^n) \equiv \mathrm{tr}_{V_N}(M^n\, v) \ \in \mathcal{M}, \tag{1-63}$$

where the trace is taken over the defining representation V_N, and

$$v = \pi(q^{-2H_\rho}) \tag{1-64}$$

is a numerical matrix which satisfies $S^2(r) = v^{-1} r v$ for the generator r of $\mathcal{G}_V$. These elements c_n are independent for $n = 1, 2, ..., rank(G)$. A proof of centrality can be found e.g. in the book [41], see also Section 1.9.4. Here we verify only invariance under (1-59):

$$\begin{aligned} c_n \to \quad & \mathrm{tr}_q(r^{-1} M^n r) = (r^{-1})^i_j (M^n)^j_k r^k_l v^l_i \\ = \quad & S(r^i_j)(M^n)^j_k v^k_l S^2(r^l_i) = (M^n)^j_k v^k_l S(S(r^l_i) r^i_j) = (M^n)^j_k v^k_j = c_n \end{aligned} \tag{1-65}$$

as required. As we shall see, these c_n for $n = 1, ...rank(G) - 1$ determine the position of the branes on the group manifold: They are quantum analogs of the s_n (1-12).

There should be another central term, which is the quantum analog of the ordinary determinant. It is known as the quantum determinant, $\det_q(M)$ (which is of course different from the quantum determinant of $Fun_q(G)$). While it can be expressed as a polynomial in c_n's ($n = 1, ..., rank(G)$), $\det_q(M)$ is invariant under the full chiral symmetry algebra. Hence we impose the constraint

$$1 = \det_q(M). \tag{1-66}$$

For other groups such as $SO(N)$ and $SP(N)$, additional constraints (which are also invariant under the full chiral quantum algebra) must be imposed. These are known and can be found in the literature [59], but their explicit form is not needed for the forthcoming considerations. Section 1.9.3 contains details about how to calculate $\det_q(M)$ and provides some explicit expressions.

1.5.3 Realizations of RE

In this section we find realizations of the RE (1-53) in terms of other known algebras. It can be viewed as an intermediate step towards finding representations. We use a technique generating new solutions out of constant solutions (trivial representations). Thus first we consider

$$R_{21}M_1^{(0)}R_{12}M_2^{(0)} = M_2^{(0)}R_{21}M_1^{(0)}R_{12}. \tag{1-67}$$

where the entries of the matrices $M^{(0)}$ are c-numbers. Then one easily checks that

$$M = L^+M^{(0)}S(L^-). \tag{1-68}$$

solves (1-53) if $L^\pm$ respects

$$RL_2^\pm L_1^\pm = L_1^\pm L_2^\pm R, \quad RL_2^+L_1^- = L_1^-L_2^+R. \tag{1-69}$$

In fact this is the same calculation as checking $\mathcal{G}_L \otimes^{\mathcal{R}} \mathcal{G}_R$ invariance of the RE. Notice that $\det_q(M) = \det_q(M^{(0)})$, due to chiral invariance of the q-determinant. Thus we have trade our original problem to the problem of finding matrices $L^\pm$ respecting (1-69). Of course there is a well-known answer due to the famous work of Faddeev, Reshetikhin and Takhtajan [24]: the relations (1-69) are exactly the same as those of the generators of $U_q(\mathfrak{g})$, (1-27). This implies that there is an algebra map

$$\begin{aligned} \mathcal{M} \quad &\rightarrow U_q(\mathfrak{g}), \\ M^i_j \quad &\mapsto (L^+M^{(0)}S(L^-))^i_j \end{aligned} \tag{1-70}$$

determined by the constant solution $M^{(0)}$, which will be very useful below. However, this map is not an isomorphism.

From now on, we will concentrate on the case where $\mathcal{M}$ is realized as a sub-algebra of $U_q(\mathfrak{g})$ via (1-70). The sub-algebra depends of course on $M^{(0)}$. We will not discuss the most general $M^{(0)}$ here (see e.g. [57, 27, 58]), but consider only the most obvious solution, which is a diagonal matrix. The specific values of the diagonal entries do not change the algebra generated by the elements of M. To be explicit, we give the solution for $\mathfrak{g} = sl(2)$ and $M^{(0)} = \mathrm{diag}(1, 1)$:

$$M = L^+S(L^-) = \begin{pmatrix} q^H & q^{-\frac{1}{2}}\lambda q^{H/2}X^- \\ q^{-\frac{1}{2}}\lambda X^+ q^{H/2} & q^{-H} + q^{-1}\lambda^2X^+X^- \end{pmatrix} \tag{1-71}$$

where $\lambda = q - q^{-1}$. One can verify that $\det_q(M) = 1$, according to (1-129). As we will see, choosing a definite representation of $U_q(\mathfrak{g})$ then corresponds to choosing a brane configuration, and determines the algebra of functions on the brane.

The other, non-diagonal solutions for $M^{(0)}$'s presumably also correspond to some branes. We hope to come back to this subject in a future paper.

1.5.4 Covariance

We show in Section 1.9.4 that for any solutions of the form $M = L^+ M^{(0)} S(L^-)$ where $M^{(0)}$ is a constant solution of RE, the "vector" rotations (1-62) can be realized as quantum adjoint action:

$$u \triangleright M^i_j = \pi^i_k(Su_1) M^k_l \pi^l_j(u_2) = u_1 \; M^i_j \; Su_2 \quad \in \mathcal{M} \tag{1-72}$$

for $u \in \mathcal{M}$, where $\pi()$ is the defining representation V_N of $U_q(\mathfrak{g})$. Here we consider $\mathcal{M} \subset U_q(\mathfrak{g})$, so that $\Delta(u) = u_1 \otimes u_2$ is defined in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$; nevertheless the rhs is in $\mathcal{M}$. The proof for $M^{(0)} = \mathbf{1}$ is very simple and well-known. This is as it should be in a quantum theory: the action of a symmetry is implemented by a conjugation in the algebra of operators. It will be essential later to perform the harmonic analysis on the branes.

1.5.5 Reality structure

An algebra $\mathcal{M}$ can be considered as a quantized (algebra of complex-valued functions on a space only if it is equipped with a $*$-structure, i.e. an anti-linear (anti)-involution. For classical unitary matrices, the condition would be $M^\dagger = M^{-1}$. To find the correct quantum version is a bit tricky here, and strictly speaking the star given below can only be justified after going to representations[6]. We determine it by requiring that on finite-dimensional representations of $M = L^+ S L^-$, the $*$ will become the usual matrix adjoint. In term of the generators of $U_q(\mathfrak{g})$, this means that $(X_i^\pm)^* = X_i^\mp$, $H_i^* = H_i$. In the $SU(2)$ case, this leads to

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a^{-1} & -qca^{-1} \\ -qa^{-1}b & q^2 d + a - q^2 a^{-1} \end{pmatrix}; \tag{1-73}$$

a^{-1} indeed exists on the irreps of $\mathcal{M}$ considered here. A closed form for this star structure for general $\mathfrak{g}$ is given in Section 1.9.1.

1.6 Representations of $\mathcal{M}$ and quantum D–branes

$\mathcal{M}$ should be considered as quanization of the manifold G in the spirit of non-commutative geometry. However, we are interested here in the quantization of the orbits $\mathcal{C}(t_\lambda)$, which are submanifolds of G. We will now explain how these arise in terms of $\mathcal{M}$, and then show that the functions on these submanifolds decompose into the same harmonics as on the classical $\mathcal{C}(t_\lambda)$, under the action of the vector symmetry algebra (1-72). This correspondence hold only up to some cutoff. In other words, the harmonic analysis on the classical and quantum $\mathcal{C}(t_\lambda)$ turns out to be the same up to the cutoff.

[6] so that certain inverses etc. exist; this This will be understood implicitly

Submanifolds in noncommutative geometry. To clarify the situation, we recall the definition of a submanifold in (non)commutative geometry. A submanifold $V \subset W$ of a classical manifold W is characterized by an embedding map

$$\iota : \quad V \hookrightarrow W \tag{1-74}$$

which is injective. Dualizing this, one obtains a surjective algebra map

$$\iota^* : \quad \mathcal{W} \to \mathcal{V}, \tag{1-75}$$

where $\mathcal{V} \cong \mathcal{W}/\mathcal{I}$ where $\mathcal{I} = \mathrm{Ker}(\iota^*)$ is a 2-sided ideal. Clearly $\mathcal{V}$ describes the internal structure of the submanifold, while $\mathcal{I}$ describes the embedding, hence the location of V in W. Explicitly,

$$\mathcal{I} = \{f : W \to \mathbb{C}; \;\; f(V) = 0\} \tag{1-76}$$

in the classical case. For noncommutative spaces, we define subspaces as quotients of the noncommutative algebra of functions by some 2-sided ideal $\mathcal{I}$, which describes the location of the subspaces.

1.6.1 Fuzzy D_λ

To identify the branes, we should therefore look for algebra maps $\mathcal{M} \to \mathcal{V}$ with nontrivial kernel and image. There are obvious candidates for such maps, given by the (irreducible) representations

$$\mathcal{M} \to U_q(\mathfrak{g}) \to Mat(V_\lambda) \tag{1-77}$$

of $U_q(\mathfrak{g})$, extending the map (1-70) for $M^{(0)} = \mathbf{1}$. On such an irreducible representation, $\mathcal{M}$ becomes the matrix algebra $Mat(V_\lambda)$, and the Casimirs $c_n = \mathrm{tr}_q(M^n)$ (1-63) of $\mathcal{M}$ take distinct values $c_n(\lambda)$ which can be calculated. This defines the ideal $\mathcal{I} = \cap_n \langle c_n - c_n(\lambda)\rangle_{\mathcal{M}}$. Hence every representation of $U_q(\mathfrak{g})$ defines a quantum submanifold. Here we consider only the realization $M = L^+SL^-$ (1-68). Then the Casimirs c_n are invariant under (vector) rotations as shown in (1-65). In view of their form, this suggests that the matrix algebra $Mat(V_\lambda)$ should be considered as quantization of (the algebra of functions on) some untwisted D–brane, the position of which is determined by the value $c_n(\lambda)$. One should realize, however, that there exist other representations of $\mathcal{M}$ which are *not* representations of $U_q(\mathfrak{g})$, which may describe different branes. We shall not consider such possibilities here.

Now we need the representation theory of $U_q(\mathfrak{g})$, which is largely understood, although quite complicated at roots of unity. Not all representations of $U_q(\mathfrak{g})$ define admissible branes. For example, they should be $\star$-representations of $\mathcal{M}$ with respect to a suitable star structure; this is because we want to describe real manifolds with complex-valued functions. We claim that the admissible representations are those V_λ with $\lambda \in P_k^+$, because they have the following properties (see Section 1.4.2):

- they are unitary, i.e. $*$ -representations of $\mathcal{M}$ with respect to the $*$ structure of Section 1.5.5 (see [54])
- their quantum-dimension $\dim_q(V_\lambda) = \mathrm{tr}_{V_\lambda}(q^{2H_\rho})$ given by (1-31) is positive
- λ corresponds precisely to the integrable modules of the affine Lie algebra $\widehat{\mathfrak{g}}$ which governs the CFT.

We will show that these irreps of $\mathcal{M}$ describe precisely the *stable* D-branes of string theory. Since the algebra $\mathcal{M}$ is[7] the direct sum of the corresponding representations, the whole group manifold is recovered in the limit $k \to \infty$ where the branes become dense. To confirm this interpretation, we will calculate the position of the branes on the group manifold, and study their geometry by performing the harmonic analysis on the branes, i.e. by determining the set of harmonics. The representations belonging to the boundary of P_k^+ will correspond to the degenerate branes.

To summarize, we propose that **the representation (1-77) of $\mathcal{M}$ for $\lambda \in P_k^+$ is a quantized or "fuzzy" D–brane, denoted by** D_λ. It is an algebra of maps from V_λ to V_λ which transforms under the quantum adjoint action (1-72) of $U_q(\mathfrak{g})$. For "small" weights[8] λ, this algebra coincides with $Mat(V_\lambda)$. There are some modifications for "large" weights λ because q is a root of unity, which will be discussed in Section 1.6.4. The reason is that $Mat(V_\lambda)$ then contains unphysical degrees of freedom which should be truncated. Moreover, we claim that the D_λ **correspond precisely to the stable D-branes on G.**

A first support for this claim is that there is indeed a one–to–one correspondence between the (untwisted) branes in string theory and these quantum branes D_λ, since both are labeled by $\lambda \in P_k^+$. To give a more detailed comparison, we calculate the traces (1-63), derive a characteristic equation, and then perform the harmonic analysis on D_λ. Furthermore, the energy (1-16) of the branes in string theory will be recovered precisely in terms of the quantum dimension.

1.6.2 Location of D_λ and values of the Casimirs

The values of the Casimirs c_n on D_λ are calculated in Section 1.9.5:

$$c_1(\lambda) = \mathrm{tr}_{V_N}\,(q^{2(H_\rho+H_\lambda)}), \tag{1-78}$$

$$c_n(\lambda) = \sum_{\nu\in V_N;\, \lambda+\nu\in P_k^+} q^{2n((\lambda+\rho)\cdot\nu-\lambda_N\cdot\rho)}\,\frac{\dim_q(V_{\lambda+\nu})}{\dim_q(V_\lambda)}, \quad n \geq 1. \tag{1-79}$$

Here λ_N is the highest weight of the defining representation V_N, and the sum in (1-79) goes over all $\nu \in V_N$ such that $\lambda+\nu$ lies in P_k^+.

The value of $c_1(\lambda)$ agrees precisely with the corresponding value (1-12) of s_1 on the classical conjugacy classes $\mathcal{C}(t_\lambda)$. For $n \geq 2$, the values of $c_n(\lambda)$ agree only approximately with s_n on $\mathcal{C}(t_\lambda)$, more precisely they agree if $\frac{\dim_q(V_{\lambda+\nu})}{\dim_q(V_\lambda)} \approx 1$, which holds provided λ is large (hence k

[7] more precisely the semi-simple quotient of $\mathcal{M}$, see Section 1.6.4
[8] see Section 1.6.4

must be large too). In particular, this holds for branes which are not “too close” to the unit element. This slight discrepancy for small λ is perhaps not too surprising, since the higher–order Casimirs are defined in terms of non-commutative coordinates and are therefore subject to operator–ordering ambiguities.

Finally, we show in Section 1.9.6 that the generators of $\mathcal{M}$ satisfy the following characteristic equation on D_λ:

$$P_\lambda(M) = \prod_{\nu \in V_N} (\ M - q^{2(\lambda+\rho)\cdot\nu - 2\lambda_N\cdot\rho}) = 0. \tag{1-80}$$

Here the usual matrix multiplication of the M^i_j is understood. Again, this (almost) matches with the classical version (1-13).

Hence we see that the positions and the “size” of the branes essentially agree with the results from string theory. In particular, their size shrinks to zero if λ approaches a corner of P_k^+, as can be seen easily in the $SU(2)$ case [45]: as λ goes from 0 to k, the branes start at the identity e, grow up to the equator, and then shrink again around $-e$. We will see that the algebra of functions on D_λ precisely reflects this behavior; however this is more subtle and will be discussed below. All of this is fundamentally tied to the fact that q is a root of unity.

It is worth emphasizing here that the agreement of the values of c_n with their classical counterparts (1-12) shows that the M’s are very reasonable variables to describe the branes.

1.6.3 Energy of D_λ and quantum dimension

Following [45], the M^i_j’s can be thought of as some matrices (as in Myers model [9]) out of which one can form an action which is invariant under the relevant quantum groups. The action should have the structure $S = \mathrm{tr}_q(1 + ...)$, where dots represent some expressions in the M’s. The point of [45] was that for some equations of motion, the ”dots”- terms vanish on classical configurations. We postulate that the equations of motion for M are given by RE (1-53). If so, then their energy is equal to

$$E = \mathrm{tr}_q(1). \tag{1-81}$$

This energy is not just a constant as might be suggested by the notation, but it depends on the representation V_λ of the algebra, where it becomes the quantum dimension (1-31)

$$\dim_q(V_\lambda) = \mathrm{tr}_q(1) = \mathrm{tr}_{V_\lambda}(q^{2H_\rho}) = \prod_{\alpha>0} \frac{\sin(\pi\frac{\alpha\cdot(\lambda+\rho))}{k+g^\vee})}{\sin(\pi\frac{\alpha\cdot\rho}{k+g^\vee})} = E_\lambda. \tag{1-82}$$

The equality follows from Weyl character formula. This is indeed the value of the energy of the corresponding D-brane in the BCFT description, (1-16).

1.6.4 The space of harmonics on D_λ.

As discussed in Section 1.2.2, we must finally match the space of functions or harmonics on D_λ with the ones on $\mathcal{C}(t_\lambda)$, up to some cutoff. Using covariance (1-72), this amounts to calculating the decomposition of the representation $Mat(V_\lambda)$ of $\mathcal{M}$ under the quantum adjoint action

of $U_q(\mathfrak{g})$ (1-62) for $\lambda \in P_k^+$. However, recall from Section 1.4.2 that the full tensor product $Mat(V_\lambda) = V_\lambda \otimes V_\lambda^*$ is problematic since q is a root of unity. To simplify the analysis, we assume first that λ is not too large[9], so that this tensor product is completely reducible. Then D_λ coincides with the matrix algebra acting on V_λ,

$$D_\lambda \cong Mat(V_\lambda) = V_\lambda \otimes V_\lambda^* \cong \oplus_\mu N_{\lambda\lambda^+}^\mu \; V_\mu, \tag{1-83}$$

where $N_{\lambda\lambda^+}^\mu$ are the usual fusion rules of $\mathfrak{g}$ which can be calculated explicitly using formula (1-119). Here λ^+ is the conjugate weight to λ, so that $V_\lambda^* \cong V_{\lambda^+}$. This has a simple geometrical meaning if μ is small enough (smaller than all *nonzero* Dynkin labels of λ, roughly speaking; see Section 1.9.2 for details): then

$$N_{\lambda\lambda^+}^\mu = mult_{\mu^+}^{(K_\lambda)} \tag{1-84}$$

where $K_\lambda \subset G$ is the stabilizer group[10] of λ, and $mult_{\mu^+}^{(K_\lambda)}$ is the dimension of the subspace of V_μ^* which is invariant under K_λ. This is proved in Section 1.9.2. Note in particular that the mode structure (for small μ) does not depend on the particular value of λ, only on its stabilizer K_λ. Comparing this with the decomposition (1-9) of $\mathcal{F}(\mathcal{C}(t_\lambda))$, we see that indeed

$$\boxed{D_\lambda \cong \mathcal{F}(\mathcal{C}(t'_\lambda))} \tag{1-85}$$

up to some cutoff in μ, where $t'_\lambda = \exp(2\pi i \frac{H_\lambda}{k+g^\vee})$. This differs slightly from (1-10), by a shift $\lambda \to \lambda + \rho$. It implies that degenerate branes do occur in the our quantum algebraic description, because λ may be invariant under a nontrivial subgroup $K_\lambda \neq T$. These degenerate branes have smaller dimensions than the regular ones. An example for this is fuzzy $\mathbb{C}P^N$, which will be discussed in some detail below. We want to emphasise that it is only the result (1-85) which allows to identify these quantized spaces with classical ones.

Here we differ from [19] who identify only regular D–branes in the CFT description, arguing that $\lambda + \rho$ is always regular. This is due to a particular limiting procedure for $k \to \infty$ which was chosen in [19]. We assume k to be large but finite, and find that degenerate branes do occur. This is in agreement with the CFT description of harmonics on D_λ, as will be discussed below. Also, note that (1-85) reconciles the results (1-78), (1-12) on the position of the branes with their mode structure as found in CFT.

Now we consider the general case where the tensor product $Mat(V_\lambda) = V_\lambda \otimes V_\lambda^*$ may not be completely reducible. Then $Mat(V_\lambda) = V_\lambda \otimes V_\lambda^*$ contains non-classical representations with vanishing quantum dimension, which have no obvious interpretation. However, there is a well–known remedy: one can replace the full tensor product by the so-called "truncated tensor product" [49], which amounts to discarding[11] the representations with $dim_q = 0$. This gives a decomposition into irreps

$$D_\lambda \cong V_\lambda \overline{\otimes} V_\lambda^* \cong \oplus_{\mu \in P_k^+} \overline{N}_{\lambda\lambda^+}^\mu \; V_\mu \tag{1-86}$$

[9] roughly speaking if $\lambda = \sum n_i \Lambda_i$, then $\sum_i n_i < \frac{1}{2}(k + g^\vee)$.

[10] which acts by the (co)adjoint action on weights

[11] note that the calculation of the Casimirs in Section 1.6.2 is still valid, because V_λ is always an irrep

involving only modules V_μ of positive quantum dimension. These $\overline{N}_{\lambda\lambda^+}^{\mu}$ are known to coincide with the fusion rules for integrable modules of the affine Lie algebra $\widehat{\mathfrak{g}}$ at level k, and can be calculated explicitly. These fusion rules in turn coincide (see e.g. [19]) with the multiplicities of harmonics on the D-branes in the CFT description, i.e. the primary (boundary) fields.

We conclude that the structure of harmonics on D_λ, (1-86) is in complete agreement with the CFT results. Moreover, it is known (see also [19]) that the structure constants of the corresponding boundary operators are essentially given by the $6j$ symbols of $U_q(\mathfrak{g})$, which in turn are precisely the structure constants of the algebra of functions on D_λ, as explained in the next subsection. Therefore our quantum algebraic description not only reproduces the correct set of boundary fields, but also essentially captures their algebra in (B)CFT.

Finally, it is interesting to note that branes D_λ which are "almost" degenerate (i.e. for λ near some boundary of P_k^+) have only few modes μ in some directions[12] and should therefore be interpreted as degenerated branes with "thin", but finite walls. They interpolate between branes of different dimensions.

1.6.5 $6j$ symbols and the algebra on D_λ

Finally we show that the structure constants of the algebra of functions on D_λ coincide precisely with those of the (twisted) algebra of boundary operators (1-19) on the branes, which are given by the $6j$ symbols of $U_q(\mathfrak{g})$. This is done using the explicit realization of D_λ as matrix algebra $Mat(V_\lambda)$, or more precisely $D_\lambda \cong V_\lambda \overline{\otimes} V_\lambda^*$ (1-86). For simplicity, we assume that $\mathfrak{g} = su(2)$ here, but all arguments generalize to the general case.

Let us denote the branes on $SU(2) \cong S^3$ as $\mathcal{S}_\lambda^2 = D_\lambda$, being 2-spheres. The decomposition (1-83) of $\mathcal{S}_\lambda^2$ is then explicitly

$$\mathcal{S}_\lambda^2 = D_\lambda = (1) \oplus (3) \oplus (5) \oplus ... \oplus (2N+1). \tag{1-87}$$

Here (k) denotes the k-dimensional representation of $su(2)$.

Let $Y_i^I \in Mat(V_\lambda)$ be a "function" on $\mathcal{S}_\lambda^2$, which transforms in the spin I representation[13] of $U_q(su(2))$. Denote with $\pi(Y_i^I)_s^r$ the matrix which represents Y_i^I on V_λ, in a weight basis of V_λ. Because it transforms in the adjoint, it must be proportional to the Clebsch–Gordan coefficient of the decomposition $(2I+1) \otimes V_\lambda \to V_\lambda$, after lowering the index r. Hence in a suitable normalization of the basis Y_i^I, we can write[14].

$$\pi(Y_i^I)_s^r = (g_q^{(N/2)})^{rr'} \begin{bmatrix} N/2 & I & N/2 \\ r' & i & s \end{bmatrix}_q = \begin{bmatrix} I & N/2 & N/2 \\ i & s & r' \end{bmatrix}_q (g_q^{(N/2)})^{r'r}. \tag{1-88}$$

[12]this is just the condition on μ discussed before (1-84)
[13]for higher groups, it will carry an additional degeneracy label
[14]recall the convention of Section 1.3 that the weight λ corresponds to $\frac{N}{2}$

Therefore the matrix representing the operator $Y_i^I Y_j^J$ is given by

$$\begin{aligned}
\pi(Y_i^I)^r_s\ \pi(Y_j^J)^s_t &= (g_q^{(N/2)})^{rr'} \begin{bmatrix} N/2 & I & N/2 \\ r' & i & s \end{bmatrix}_q (g_q^{(N/2)})^{ss'} \begin{bmatrix} N/2 & J & N/2 \\ s' & j & t \end{bmatrix}_q \\
&= \sum_K \begin{Bmatrix} N/2 & J & N/2 \\ I & N/2 & K \end{Bmatrix}_q \begin{bmatrix} I & J & K \\ i & j & k' \end{bmatrix}_q (g_q^{(K)})^{k'k} \begin{bmatrix} K & N/2 & N/2 \\ k & t & r' \end{bmatrix}_q (g_q^{(N/2)})^{r'r} \\
&= \sum_K \begin{Bmatrix} I & J & K \\ N/2 & N/2 & N/2 \end{Bmatrix}_q \begin{bmatrix} I & J & K \\ i & j & k' \end{bmatrix}_q (g_q^{(K)})^{k'k}\ \pi(Y_k^K)^r_t.
\end{aligned} \tag{1-89}$$

Here we used the identity

$$\begin{Bmatrix} N/2 & J & N/2 \\ I & N/2 & K \end{Bmatrix}_q = \begin{Bmatrix} I & J & K \\ N/2 & N/2 & N/2 \end{Bmatrix}_q, \tag{1-90}$$

which is proved in [60]. This calculation is represented graphically in Figure 1.3, which shows that it simply boils down to the definition of the $6j$–symbols. Associativity could also be verified using the q-Biedenharn Elliott identity [60]. Therefore the algebra of $\mathcal{S}^2_\lambda$ is precisely (1-19), which is a twist of the algebra (1-18).

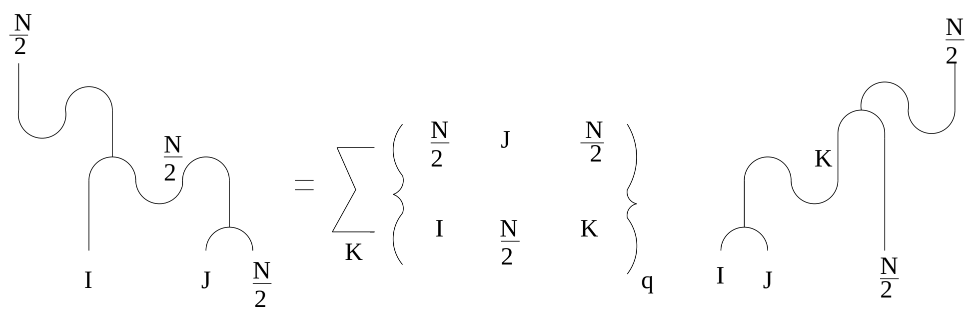

Figure 1.3: Derivation of the algebra (1-19)

1.7 Examples

1.7.1 Fuzzy $\mathbb{C}P_q^{N-1}$

Particularly interesting examples of degenerate conjugacy classes are the complex projective spaces $\mathbb{C}P^{N-1}$. We shall demonstrate the scope of our general results by extracting some explicit formulae for this special case. This gives a q-deformation of the fuzzy $\mathbb{C}P^{N-1}$ discussed in [46, 47].

We first give a more explicit description of branes on $SU(N)$. Let us parameterize the matrix M ($= L^+SL^-$ acting on V_λ) as

$$M = \xi_\alpha \lambda^\alpha = \sum_a \xi_a \lambda^a + \xi_0 \lambda^0 \tag{1-91}$$

where $\alpha = (0, a)$ and $a = 1, 2, ..., N^2 - 1$. The ξ_α will be generators of a non-commutative algebra, and $\lambda^a = (\lambda^a)^\alpha{}_{\dot\beta}$ for $a = 1, 2, .., N^2 - 1$ are q-deformed Gell-Mann matrices; we set $\lambda^0 \equiv \mathbf{1}$. Using covariance (1-72), M transforms as

$$M \to u \triangleright M = \xi_\alpha \; \pi(Su_1)\lambda^\alpha \pi(u_2) = (u_1 \xi_\alpha S u_2) \; \lambda^a. \tag{1-92}$$

We will hence choose a basis such that

$$\xi_a \to u \triangleright \xi_a = u_1 \xi_a S u_2 = \xi_b \; (\pi_{(ad)}(u))^a_b \tag{1-93}$$

transforms under the adjoint of $U_q(su(N)_V)$, and

$$\pi(Su_1)\lambda^a \pi(u_2) = (\pi_{(ad)}(u))^a_b \; \lambda^b \tag{1-94}$$

can be viewed as the right adjoint action of $U_q(su(N)_V)$ on $Mat(N, \mathbb{C})$. Therefore the λ^a are the intertwiners $(N) \otimes (\overline{N}) \to (N^2 - 1)$ under the right action of $U_q(su(N)_V)$. They satisfy the relation

$$\lambda^a \lambda^b = \frac{1}{\dim_q(V_N)} g^{ab} + (d^{ab}{}_c + f^{ab}{}_c)\lambda^c \tag{1-95}$$

where g^{ab}, $d^{ab}{}_c$ and $f^{ab}{}_c$ are (right) invariant tensors in a suitable normalization, and $\mathrm{tr}_q(\lambda^a) = 0$ (for $a \neq 0$). We can now express the Casimirs c_n (1-79) in terms of the new generators:

$$\begin{aligned} c_1 &= \mathrm{tr}_q(M) = \xi_0 \dim_q(V_N), && \text{(1-96)} \\ c_2 &= g^{ab} \; \xi_a \xi_b + \xi_0^2 \dim_q(V_N), && \text{(1-97)} \\ & && \text{(1-98)} \end{aligned}$$

etc, which are numbers on each D_λ. An immediate consequence of (1-96) is

$$[\xi_0, \xi_a] = 0 \tag{1-99}$$

for all a. One can show furthermore that the reflection equation (1-53), which is equivalent to the statement that the (q-)antisymmetric part of MM vanishes, implies that

$$f^{ab}{}_c \; \xi_a \xi_b = \alpha \; \xi_0 \xi_c. \tag{1-100}$$

On a given brane D_λ, ξ_0 is a number determined by (1-96), while α is a (universal) constant which can be determined explicitly, as indicated below.

(1-99) and (1-100) hold for all branes D_λ. Now consider $\mathbb{C}P^{N-1} \cong SU(N)/U(N-1)$, which is the conjugacy class through $\lambda = n\Lambda_1$ (or equivalently $\lambda = n\Lambda_N$) where Λ_i are the

fundamental weights; indeed, the stabilizer group for $n\Lambda_1$ is $U(N-1)$. The quantization of $\mathbb{C}P^{N-1}$ is therefore the brane D_λ. It is characterized by a further relation among the generators ξ_a, which has the form

$$d^{ab}{}_c\, \xi_a \xi_b = \beta_n\, \xi_c \tag{1-101}$$

where the number β_n can be determined explicitly as indicated below. For $q=1$, these relations reduce to the ones given in [46]. (1-101) can be quickly derived using the results in Section 1.6.4: It is easy to see using (1-119) that

$$D_{n\Lambda_1} \cong \oplus_n (n, 0, ..., 0, n) \tag{1-102}$$

up to some cutoff, where $(k_1, ..., k_N)$ denotes the highest-weight representation with Dynkin labels $k_1, ..., k_N$. The important point is that all multiplicities are one. It follows that the function $d^{ab}{}_c \xi_a \xi_b$ on $D_{n\Lambda_1}$ must be proportional to ξ_c, because it transforms as $(1, 0, ..., 0, 1)$ (which is the adjoint). Hence (1-101) holds.

The constant α in (1-100) can be calculated either by working out RE explicitly, or by specializing (1-100) for D_{Λ_1}. We shall only indicate this here: On D_{Λ_1}, $\xi_a = c\lambda_a$ for some $c \in \mathbb{C}$. Plugging this into (1-100), one finds $c\, f^{ab}{}_c\, \lambda_a \lambda_b = \alpha\, \xi_0 \lambda_c$, and $c^2\, g^{ab}\, \lambda_a \lambda_b + \xi_0^2\, \dim_q(V_N) = c_2$. Calculating ξ_0 and the Casimirs explicitly on D_{Λ_1}, one obtains α which vanishes as $q \to 1$. Similarly using the explicit value of c_3 given in Section 1.6.2, one can also determine β_n. Alternatively, they be calculated using creation - and annihilation operator techniques of [61], [36, 62].

In any case, we recover the relations of fuzzy $\mathbb{C}P^{N-1}$ as given in [46] in the limit $q \to 1$. As an algebra, it is in fact identical to it, as long as k is sufficiently large.

1.7.2 $G = SU(2)$ model

In this section we shall show how one can recover the results of [45] from the general formalism we discussed so far. The solution to RE given by $L^\pm$ operators and $M^{(0)} = \mathrm{diag}(1,1)$ is

$$M = L^+ M^{(0)} S(L^-) = \begin{pmatrix} q^H & q^{-\frac{1}{2}} \lambda q^{H/2} X^- \\ q^{-\frac{1}{2}} \lambda X^+\, q^{H/2} & q^{-H} + q^{-1}\lambda^2 X^+ X^- \end{pmatrix} \tag{1-103}$$

Let us parameterize the M matrix as

$$M = \begin{pmatrix} M_4 - iM_0 & -iq^{-3/2}\sqrt{[2]}M_+ \\ iq^{-1/2}\sqrt{[2]}M_- & M_4 + iq^{-2}M_0 \end{pmatrix} \tag{1-104}$$

as in (1-91), then RE is equivalent to

$$[M_4, M_l] = 0, \quad \epsilon_l^{ij}\ M_i M_j = i(q - q^{-1}) M_4 M_l \tag{1-105}$$

which will be studied in great detail in Section ... . In order to calculate the central terms we need

$$v = \pi(q^{-2H_\rho}) = \pi(q^{-H}) = \mathrm{diag}(q^{-1}, q) \tag{1-106}$$

so that using (1-63),(1-129)

$$
\begin{aligned}
c_1 &= \mathrm{tr}_q(M) = [2]\, M_4 && (1\text{-}107)\\
c_2 &= \mathrm{tr}_q(M^2) = [2]\ (M_4^2 - q^{-2} g^{ij} M_i M_j) && (1\text{-}108)\\
\det_q(M) &= M_4^2 + M_0^2 - q^{-1} M_+ M_- - q\, M_- M_+ = M_4^2 + g^{ij} M_i M_j. && (1\text{-}109)
\end{aligned}
$$

Only $\det_q(M)$ is invariant under $U_q(su(2)_L \times su(2)_R)_{\mathcal{R}}$. The explicit value of $M_4 = c_1/[2]$ is obtained from

$$M_4 = \frac{1}{[2]}(q^{H-1} + q^{-(H-1)} + \lambda^2 X^+ X^-)$$

which is proportional to the standard Casimir of $U_q(su(2))$. On the n-th brane D_n, H takes the value $-n$ on the lowest weight vector, thus

$$\boxed{M_4 = \cos(\tfrac{(n+1)\pi}{k+2}) / \cos(\tfrac{\pi}{k+2})} \qquad (1\text{-}110)$$

for $n = 0, 1, .., k$. This leads to the pattern of branes as shown in Figure 1.4. If the square of the radius of the quantum S^3 is chosen to be $\det_q(M) = k$ (which is the value given by the supergravity solution for the background), $g^{ij} M_i M_j$ leads to the correct formulae for the square of the radius of the n-th branes.

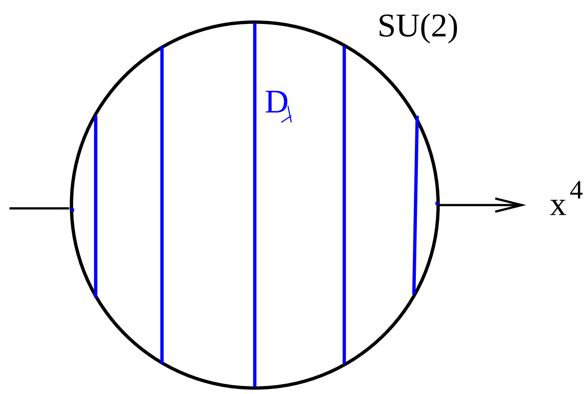

Figure 1.4: Position of branes on $SU(2)$

1.8 Summary

We proposed in this chapter a simple and compact description of all (untwisted) D-branes on group manifolds G based on the reflection equation RE. The model can be viewed as a finite matrix model in the spirit of the non-abelian DBI model of D0-branes [9], but contrary to the latter it yields results well beyond the $1/k$ approximation. In fact, the model properly describes all branes on the group manifold regardless of their positions. This covers an astonishing wealth

of data on the configurations and properties of branes such as their positions and spaces of functions, which are shown to be in very good agreement with the CFT data. It also shows that M is a very reasonable variable to describe the branes. Our construction also sheds light on the fact that the energies of these branes are given by so-called quantum dimensions.

The branes are uniquely given by certain "canonical" irreps of the RE algebra, and their world-volume can be interpreted as quantum manifolds. The characteristic feature of our construction is the covariance of RE under a quantum analog of the group of isometries $G_L \times G_R$ of G. A given brane configuration breaks it to the diagonal (quantum) $\mathcal{G}_V$, an analog of the classical vector symmetry G_V.

It should be clear to the reader, however, that the present picture does not cover all aspects of branes physics on group manifolds. For example, we did not study all representations of RE, only the most obvious ones which are induced by the algebra map $RE \to U_q(\mathfrak{g})$. There exist other representations of RE, some of which can be investigated using the technique in Section 1.5.3, some of which may be entirely different. One may hope that all of the known D-branes on groups, including those not discussed here such as twisted branes or "type B"-branes, can be described in this way. This is an interesting open problem, in particular in view of the recent progress [14] made on the string-theoretic side of twisted branes. Moreover, we did not touch here the dynamical aspects of D-branes, such as their excitations and interactions. For this it may be necessary to extend the algebraic content presented here, and the well-developed theory of quantum groups may become very useful.

These considerations have a large number of possible physical applications, independent of string theory. The general construction of quantized branes presented here provides a variety of specify examples of finite ("fuzzy") quantum spaces, such as $\mathbb{C}P^N_q$. They may serve as useful testing grounds for noncommutative field theories, which are completely finite on these spaces, due to the finite number of degrees of freedom. This is the subject of the remaining chapters.

1.9 Technical complements to Chapter 1

1.9.1 Reality structure

In term of the generators of $U_q(\mathfrak{g})$, the star is defined as $(X_i^\pm)^* = X_i^\mp$, $H_i^* = H_i$. Using [63]

$$\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}^{-1} \tag{1-111}$$

which holds for $|q| = 1$, it follows that

$$M^\dagger := M^{T*} = SL^+SL^-M^{-1}L^+L^-. \tag{1-112}$$

The rhs is indeed in $\mathcal{M}$, because the L - matrices can be expressed on irreps in terms of the M - matrices[15]. This can be cast into a more convenient form using the generator of the "longest

[15] this holds only on completely reducible representations

Weyl reflection" $\omega \in U_q(\mathfrak{g})$ [64] (more precisely in an extension thereof), which satisfies

$$\Delta(\omega) = \mathcal{R}^{-1}\omega \otimes \omega = \omega \otimes \omega \mathcal{R}_{21}^{-1}. \tag{1-113}$$

Furthermore, it implements the automorphism $S\theta\gamma$ of $U_q(\mathfrak{g})$ as an inner automorphism:

$$\begin{aligned} \theta(X_i^{\pm}) &= X_i^{\mp},\ \theta(H_i) = H_i, \\ S\theta\gamma(u) &= \omega^{-1} u \omega \end{aligned} \tag{1-114}$$

for any $u \in U_q(\mathfrak{g})$. Hence the star can also be written as

$$M^{\dagger} = \pi(\omega^{-1})\omega^{-1}M^{-1}\omega\pi(\omega) = \pi(\omega^{-1})S\theta\gamma(M^{-1})\pi(\omega) \tag{1-115}$$

where π is the defining representation V_N. This form is useful to verify the consistency of the star with the relations (1-53) and (1-66). The classical limit $q \to 1$ of this star structure is correct, because then $\pi(\omega^{-1})S\theta\gamma(M)\pi(\omega) \to M$. In the $SU(2)$ case, one recovers the star given in (1-73).

1.9.2 The harmonics on the branes

The modes on $\mathcal{C}(t)$ (1-9). Consider the map

$$\begin{aligned} G/K_t &\to \mathcal{C}(t), \\ gK_t &\mapsto gtg^{-1} \end{aligned}$$

which is clearly well–defined and bijective. It is also compatible with the group actions, in the sense that the adjoint action of G on $\mathcal{C}(t)$ translates into the *left* action on G/K_t. Hence we want to decompose functions on G/K_t under the left action of G.

Functions on G/K_t can be considered as functions on G which are invariant under the *right* action of K_t, and this correspondence is one-to-one (because this action is free). Now the Peter-Weyl theorem states that the space of functions on G is isomorphic as a bimodule to

$$\mathcal{F}(G) \cong \bigoplus_{\lambda \in P^+} V_\lambda \otimes V_\lambda^*. \tag{1-116}$$

Here λ runs over all dominant integral weights, and V_λ is the corresponding highest-weight module. Let $mult_{\lambda^+}^{(K_t)}$ be the dimension of the subspace of $V_\lambda^* \equiv V_{\lambda^+}$ which is invariant under the action of K_t. Then

$$\mathcal{F}(\mathcal{C}(t)) \cong \bigoplus_{\lambda \in P} mult_{\lambda^+}^{(K_t)}\, V_\lambda \tag{1-117}$$

follows.

The modes on D_λ and proof of (1-84). We are looking for the Littlewood–Richardson coefficients $N_{\lambda\lambda^+}^{\mu}$ in the decomposition

$$V_\lambda \otimes V_\lambda^* \cong \oplus_\mu N_{\lambda\lambda^+}^{\mu}\, V_\mu \tag{1-118}$$

of $\mathfrak{g}$ - modules. Now we use $N_{\lambda\lambda^+}^{\mu} = N_{\lambda\mu^+}^{\lambda}$ (because $N_{\lambda\lambda^+}^{\mu}$ is given by the multiplicity of the trivial component in $V_\lambda \otimes V_{\lambda^+} \otimes V_{\mu^+}$, and so is $N_{\lambda\mu^+}^{\lambda}$). But $N_{\lambda\mu^+}^{\lambda}$ can be calculated using the formula [49]

$$N_{\lambda\mu^+}^{\lambda} = \sum_{\sigma\in W} (-1)^\sigma\; mult_{\mu^+}(\sigma\star\lambda-\lambda), \tag{1-119}$$

where W is the Weyl group of $\mathfrak{g}$. Here $mult_{\mu^+}(\nu)$ is the multiplicity of the weight space ν in V_{μ^+}, and $\sigma\star\lambda = \sigma(\lambda+\rho)-\rho$ denotes the action of σ with reflection center $-\rho$. Now one can see already that for large, generic λ (so that $\sigma\star\lambda-\lambda$ is not a weight of V_{μ^+} unless $\sigma=1$), it follows that $N_{\lambda\mu^+}^{\lambda} = mult_{\mu^+}(0) = mult_{\mu^+}^{(T)}$, which proves (1-84) for the generic case. To cover all possible λ, we proceed as follows:

Let $\mathfrak{k}$ be the Lie algebra of K_λ, and $W_\mathfrak{k}$ its Weyl group; it is the subgroup of W which leaves λ invariant, generated by those reflections which preserve λ (the $u(1)$ factors in $\mathfrak{k}$ do not contribute to $W_\mathfrak{k}$). If μ is "small enough", then the sum in (1-119) can be restricted to $\sigma\in W_\mathfrak{k}$, because otherwise $\sigma\star\lambda-\lambda$ is too large to be in V_{μ^+}; this defines the cutoff in μ. It holds for any given μ if λ has the form $\lambda = n\lambda_0$ for large $n\in\mathbb{N}$ and fixed λ_0[16]. We will show below that

$$mult_{\mu^+}^{(K_\lambda)} = \sum_{\sigma\in W_\mathfrak{k}} (-1)^\sigma\; mult_{\mu^+}(\sigma\star\lambda-\lambda) \tag{1-120}$$

for all μ, which implies (1-84). Recall that the lhs is the dimension of the subspace of V_{μ^+} which is invariant under K_λ.

To prove (1-120), first observe the following fact: Let V_λ be the highest weight irrep of some simple Lie algebra $\mathfrak{k}$ with highest weight λ. Then

$$\sum_{\sigma\in W_\mathfrak{k}} (-1)^\sigma\; mult_{V_\lambda}(\sigma\star 0) = \delta_{\lambda,0} \tag{1-121}$$

i.e. the sum vanishes unless V_λ is the trivial representation; here $\mathfrak{k}=u(1)$ is allowed as well. This follows again from (1-119), considering the decomposition of $V_\lambda\otimes(1)$. More generally, assume that $\mathfrak{k}=\oplus_i\mathfrak{k}_i$ is a direct sum of simple Lie algebras $\mathfrak{k}_i$, with corresponding Weyl group $W_\mathfrak{k}=\prod_i W_i$. Its irreps have the form $V=\otimes_i V_{\lambda_i}$, where V_{λ_i} denotes the highest weight module of $\mathfrak{k}_i$ with highest weight λ_i. We claim that the relation

$$\sum_{\sigma\in W_\mathfrak{k}} (-1)^\sigma\; mult_V(\sigma\star 0) = \prod_i\, \delta_{\lambda_i,0} \tag{1-122}$$

[16]This constitutes our definition of "classical limit". For weights λ which do not satisfy this requirement, the corresponding D–brane D_λ cannot be interpreted as "almost–classical". Here we differ from the approach in [19], which do not allow degenerate λ_0.

still holds. Indeed, assume that some $\lambda_i \neq 0$; then

$$\sum_{\sigma\in W_{\mathfrak{k}}} (-1)^\sigma \; mult_V(\sigma\star 0) = \Big(\sum_{\sigma'\in \prod' W_i} (-1)^{\sigma'}\Big)\Big(\sum_{\sigma\in W_i} (-1)^{\sigma_i} mult_V(\sigma_i\star(\sigma'\star 0))\Big) = 0$$

in self–explanatory notation. The last bracket vanishes by (1-121), since $(\sigma'\star 0)$ has weight 0 with respect to $\mathfrak{k}_i$, while V contains no trivial component of $\mathfrak{k}_i$ (notice that $\rho = \sum\rho_i$, and the operation $\star$ is defined component-wise). Therefore for any (finite, but not necessarily irreducible) $\mathfrak{k}$–module V, the number of trivial components in V is given by $\sum_{\sigma\in W_{\mathfrak{k}}} (-1)^\sigma \; mult_V(\sigma\star 0)$.

We now apply this to (1-120). Since the sum is over $\sigma\in W_{\mathfrak{k}}$, we have $\sigma(\lambda) = \lambda$ by definition, and $\sigma\star\lambda - \lambda = \sigma\star 0$. Hence the rhs can be replaced by $\sum_{\sigma\in W_{\mathfrak{k}}} (-1)^\sigma \; mult_{\mu^+}(\sigma\star 0)$. But this is precisely the number of vectors in V_{μ^+} which are invariant under K_λ, as we just proved. Notice that we use here the fact that $\mathfrak{k}$ contains the Cartan sub-algebra of $\mathfrak{g}$, so that the space of weights of $\mathfrak{k}$ is the same as the space of weights of $\mathfrak{g}$; therefore the multiplicities in (1-120) and (1-122) are defined consistently. This is why we had to include the case $\mathfrak{k}_i = su(1)$ in the above discussion.

To calculate the decomposition (1-86) for all allowed λ (with $\dim_q(V_\lambda) > 0$), the ordinary multiplicities in (1-83) should be replaced with with their truncated versions $\overline{N}_{\lambda\mu^+}^{\lambda}$ (1-86) corresponding to $U_q(\mathfrak{g})$ at roots of unity. There exist generalizations of the formula (1-119) which allow to calculate $\overline{N}_{\lambda\mu^+}^{\lambda}$ efficiently; we refer here to the literature, e.g. [19].

1.9.3 The quantum determinant

Here we quote a formula for the quantum determinant, following [59, 65]. We need the q-deformed totally (q)–antisymmetric tensor $\varepsilon_q^{i_1\dots i_N}$, which for $U_q(sl(N))$ has the form

$$\varepsilon_q^{\sigma(1)\dots\sigma(N)} = (-q)^{-l(\sigma)} = \varepsilon^q_{\sigma(1)\dots\sigma(N)} \tag{1-123}$$

where $l(\sigma)$ is the length of the permutation σ. The important formula respected by ε_q is

$$\hat{R}_{i,i+1}\varepsilon_q^{1\dots N} = -q^{-\frac{N+1}{N}} \; \varepsilon_q^{1\dots N}. \tag{1-124}$$

(for $U_q(sl(N))$. The factor differs for other groups.) The most obvious form of the determinant is

$$det(M)\varepsilon_q^{1\dots N} = (M_n)(\hat{R}_n M_n \hat{R}_n)\dots(\hat{R}_2\hat{R}_3\dots\hat{R}_n M_n\hat{R}_n\dots\hat{R}_3\hat{R}_2)\varepsilon_q^{12\dots n} \tag{1-125}$$

where $\hat{R}_k = \hat{R}_{k-1,k}$ (ignoring a possible constant factor). We check invariance inder the chiral coaction $M\to s^{-1}Mt$ (noting that $t_n\hat{R}_n s_n^{-1} = s_{n-1}^{-1}\hat{R}_n t_{n-1}$): observe first that

$$\begin{aligned}(M_n)(\hat{R}_n M_n\hat{R}_n) \;&\to\; s_n^{-1}(M_n)t_n(\hat{R}_n s_n^{-1}M_n t_n\hat{R}_n) = s_n^{-1}M_n(s_{n-1}^{-1}\hat{R}_n t_{n-1})M_n t_n\hat{R}_n \\ &=\; s_n^{-1}s_{n-1}^{-1}(M_n\hat{R}_n M_n)t_{n-1}t_n\hat{R}_n = s_n^{-1}s_{n-1}^{-1}(M_n)(\hat{R}_n M_n\hat{R}_n)t_{n-1}t_n\end{aligned}$$

let us work out one more step: since

$$t_{n-1}t_n\hat{R}_{n-1}\hat{R}_n s_n^{-1} = t_{n-1}\hat{R}_{n-1}s_{n-1}^{-1}\hat{R}_n t_{n-1} = s_{n-2}^{-1}\hat{R}_{n-1}\hat{R}_n t_{n-2}t_{n-1} \tag{1-126}$$

etc, it follows that

$$
\begin{aligned}
&(M_n)(\hat{R}_n M_n \hat{R}_n)(\hat{R}_{n-1}\hat{R}_n M_n \hat{R}_n \hat{R}_{n-1}) \to \\
&= s_n^{-1} s_{n-1}^{-1}(M_n)(\hat{R}_n M_n \hat{R}_n) t_{n-1} t_n (\hat{R}_{n-1}\hat{R}_n s_n^{-1} M_n t_n \hat{R}_n \hat{R}_{n-1}) \\
&= s_n^{-1} s_{n-1}^{-1}(M_n)(\hat{R}_n M_n \hat{R}_n) s_{n-2}^{-1} \hat{R}_{n-1}\hat{R}_n t_{n-2} t_{n-1} M_n t_n \hat{R}_n \hat{R}_{n-1} \\
&= s_n^{-1} s_{n-1}^{-1} s_{n-2}^{-1}(M_n)(\hat{R}_n M_n \hat{R}_n) \hat{R}_{n-1}\hat{R}_n M_n t_{n-2} t_{n-1} t_n \hat{R}_n \hat{R}_{n-1} \\
&= s_n^{-1} s_{n-1}^{-1} s_{n-2}^{-1}(M_n)(\hat{R}_n M_n \hat{R}_n)(\hat{R}_{n-1}\hat{R}_n M_n \hat{R}_n \hat{R}_{n-1}) t_{n-2} t_{n-1} t_n
\end{aligned}
\tag{1-127}
$$

Invariance now follows from the determinant condition $t_1 ... t_{n-1} t_n \varepsilon_q^{12...n} = \varepsilon_q^{12...n}$, and similarly for the s. This in turn implies that indeed $det(M) = 1$ (fixing the constant factor in the definition of $det(M) = 1$) in the realization $M = L^+ S L^-$.

(another possible, essentially equivalent form of the determinant is [65]

$$det(M) = (M_n \hat{R}_2 ... \hat{R}_{n-1} \hat{R}_n)^n \varepsilon^{12...n} \tag{1-128}$$

)

For $N = 2$, this becomes $det(M) = M_2 \hat{R}_{12} M_2 \hat{R}_{12}\ \varepsilon^{12} = -q^{-\frac{3}{2}} M_2 \hat{R}_{12} M_2 \varepsilon^{12}$, which is proportional to ε_q^{12} times $(M_1^1 M_2^2 - q^2 M_1^2 M_2^1)$. Hence we find the quantum determinant

$$\det_q(M) = (M_1^1 M_2^2 - q^2 M_1^2 M_2^1) \tag{1-129}$$

as in [41]. For other groups such as $SO(N)$ and $SP(N)$, the explicit form for $\varepsilon_q^{i_1 \dots i_N}$ is different, and additional constraints (which are also invariant under the chiral symmetries) must be imposed. These are known and can be found in the literature [59, 65].

1.9.4 Covariance of M and central elements

For any numerical matrix $M^{(0)}$ (in the defining representation of $U_q(\mathfrak{g})$), consider

$$M = L^+ M^{(0)} S L^- = (\pi \otimes 1)(\mathcal{R}_{21})\ M^{(0)}\ (\pi \otimes 1)\mathcal{R}_{12}. \tag{1-130}$$

Let $\mathcal{M} \subset U_q(\mathfrak{g})$ be the sub-algebra generated by the entries of this matrix. First, we note that $\mathcal{M}$ is a (left) coideal sub-algebra, which means that $\Delta(\mathcal{M}) \in U_q(\mathfrak{g}) \otimes \mathcal{M}$. This is verified simply by calculating the coproduct of M,

$$\Delta(M^i_l) = L^{+i}{}_s S L^{-t}{}_l \otimes (M)^s_t. \tag{1-131}$$

In particular if $M^{(0)}$ is a constant solution of the reflection equation (1-53), it follows by taking the defining representation of (1-130) that $[\pi(M^i_j), M^{(0)}] = 0$, and therefore $[\pi(\mathcal{M}), M^{(0)}] = 0$. Then for any $u \in \mathcal{M} \subset U_q(\mathfrak{g})$,

$$
\begin{aligned}
((\pi \otimes 1)\Delta(u))M &= (\pi \otimes 1)(\Delta(u)\mathcal{R}_{21})\ M^{(0)}\ SL^- \\
&= (\pi \otimes 1)(\mathcal{R}_{21}\Delta'(u))\ M^{(0)}\ SL^- \\
&= L^+\ M^{(0)}\ (\pi \otimes 1)(\Delta'(u)\mathcal{R}_{12}) \\
&= L^+\ M^{(0)}\ SL^-(\pi \otimes 1)\Delta(u) = M\ (\pi \otimes 1)\Delta(u).
\end{aligned}
\tag{1-132}
$$

In the second line we used $\Delta'(u) \equiv u_2 \otimes u_1 = R\Delta(u)R^{-1}$, in the third line, the coideal property (1-131). Using Hopf algebra identities (i.e. multiplying from left with $(\pi(Su_0) \otimes 1)$ and from the right with $(1 \otimes Su_3)$), this is equivalent to $(1 \otimes u_1)M(1 \otimes Su_2) = (\pi(Su_1) \otimes 1)M(\pi(u_2) \otimes 1)$, or

$$u_1 M S u_2 = \pi(Su_1) M \pi(u_2) \tag{1-133}$$

for any $u \in \mathcal{M}$, as desired. This implies immediately that

$$u_1 \mathrm{tr}_q(M^n) S u_2 = \mathrm{tr}_q(\pi(Su_1) M^n \pi(u_2)) = \varepsilon(u)\ \mathrm{tr}_q(M^n), \tag{1-134}$$

or equivalently

$$[u, \mathrm{tr}_q(M^n)] = 0 \tag{1-135}$$

for any $u \in \mathcal{M}$. This proves in particular that the Casimirs c_n (1-63) are indeed central.

1.9.5 Evaluation of Casimirs

Evaluation of c_1 Consider the fuzzy D–brane D_λ. Then c_1 acts on the highest–weight module V_λ, and has the form

$$c_1 = \mathrm{tr}_q(L^+ S L^-) = (\mathrm{tr}_q \pi \otimes 1)(\mathcal{R}_{21}\mathcal{R}_{12}). \tag{1-136}$$

Because it is a Casimir, it is enough to evaluate it on the lowest–weight state $|\lambda_-\rangle$ of V_λ, given by $\lambda_- = \sigma_m(\lambda)$ where σ_m denotes the longest element of the Weyl group. Now the universal $\mathcal{R}$ has the form

$$\mathcal{R} = q^{H_i (B^{-1})_{ij} \otimes H_j}\ (1 \otimes 1 + \sum U^+ \otimes U^-). \tag{1-137}$$

Here B is the (symmetric) matrix $d_j^{-1} A_{ij}$ where A is the Cartan Matrix, d_i are the lengths of the simple roots ($d_i = 1$ for $\mathfrak{g} = su(N)$) and U^+, U^- stands for terms in the Borel sub-algebras of rising respectively lowering operators. Hence only the diagonal elements of $(SL^-)^i_j$ are non–vanishing on a lowest–weight state, and due to the trace only the diagonal elements of $(L^+)^i_j$ enter. We can therefore write

$$c_1\ |\lambda_-\rangle = (\mathrm{tr}_q \pi \otimes 1)(q^{2\ H_i (B^{-1})_{ij} \otimes H_j})|\lambda_-\rangle = (\mathrm{tr}\ \pi \otimes 1)(q^{-2H_\rho} \otimes 1)(q^{2\ H_i (B^{-1})_{ij} \otimes H_j})\ |\lambda_-\rangle \tag{1-138}$$

Here $H_\lambda |\mu\rangle = (\lambda \cdot \mu)\ |\mu\rangle$ for any weight μ. Therefore the eigenvalue of c_1 is

$$c_1 = \sum_{\mu \in V_N} q^{-2\mu\cdot\rho + 2\mu\cdot\lambda_-} = \sum_{\mu \in V_N} q^{2\mu\cdot(-\rho + \lambda_-)}. \tag{1-139}$$

Using $\sigma_m(\rho) = -\rho$, this becomes

$$c_1 = \sum_{\mu \in V_N} q^{2(\sigma_m(\mu))\cdot(\rho+\lambda)} = \mathrm{tr}_{V_N}\ (q^{2(\rho+\lambda)}) \tag{1-140}$$

because the weights of V_N are invariant under the Weyl group.

Evaluation of c_n in general Since c_n is proportional to the identity matrix on irreps, it is enough to calculate $\mathrm{tr}_q(c_n) = \mathrm{tr}_{V_\lambda}(c_n\ q^{-2H_\rho})$ on V_λ, noting that $\mathrm{tr}_q(1) = \dim_q(V_\lambda)$ is known explicitly:

$$\mathrm{tr}_q(c_n) = (\mathrm{tr}_q \otimes \mathrm{tr}_q)((\mathcal{R}_{21}\mathcal{R}_{12})^n) \tag{1-141}$$

where the traces are over $Mat(N)$ and $Mat(V_\lambda)$. Now we use the fact that $\mathcal{R}_{21}\mathcal{R}_{12}$ commutes with $\Delta(U_q(\mathfrak{g}))$, i.e. it is constant on the irreps of $V_N \otimes V_\lambda$, and observe that $\Delta(q^{-2H_\rho}) = q^{-2H_\rho} \otimes q^{-2H_\rho}$, which means that the quantum trace factorizes. Hence we can decompose the tensor product $V_N \otimes V_\lambda$ into irreps:

$$V_N \otimes V_\lambda = \oplus_{\mu \in P_k^+} V_\mu \tag{1-142}$$

where the sum goes over all μ which have the form $\mu = \lambda + \nu$ for ν a weight of V_N. The multiplicities are equal one because V_N is the defining representation. The eigenvalues of $\mathcal{R}_{21}\mathcal{R}_{12}$ on V_μ are known [66] to be $q^{c_\mu - c_\lambda - c_{\lambda_N}}$, where λ_N denotes the highest weight of V_N and $c_\lambda = \lambda\cdot(\lambda + 2\rho)$. Now for $\mu = \lambda + \nu$,

$$c_\mu - c_\lambda - c_{\lambda_N} = 2(\lambda + \rho) \cdot \nu - 2\lambda_N \cdot \rho, \tag{1-143}$$

hence the set of eigenvalues of $\mathcal{R}_{21}\mathcal{R}_{12}$ is

$$\{q^{2(\lambda+\rho)\cdot\nu - 2\lambda_N\cdot\rho};\ \ \nu \in V_N\}. \tag{1-144}$$

Putting this together, we obtain

$$\mathrm{tr}_q(c_n) = c_n\ \mathrm{tr}_{V_\lambda}(q^{-2H_\rho}) = \sum_\mu\ q^{2n((\lambda+\rho)\cdot\nu - \lambda_N\cdot\rho)}\ \mathrm{tr}_{V_\mu}(q^{-2H_\rho}) \tag{1-145}$$

where the sum is as explained above. Then (1-79) follows, since $\mathrm{tr}_{V_\mu}(q^{-2H_\rho}) = \dim_q(V_\mu)$.

1.9.6 Characteristic equation for M.

(1-80) can be seen as follows: On D_λ, the quantum matrices M^i_j become the operators

$$(\pi^i_j \otimes \pi_\lambda)(\mathcal{R}_{21}\mathcal{R}_{12}) \tag{1-146}$$

acting on V_λ. As above, the representation of $\mathcal{R}_{21}\mathcal{R}_{12}$ acting on $V_N \otimes V_\lambda$ has eigenvalues $\{q^{c_\mu - c_\lambda - c_{\lambda_N}} = q^{2(\lambda+\rho)\cdot\nu - 2\lambda_N\cdot\rho}\}$ on V_μ in the decomposition (1-142). Here $\mu = \lambda + \nu$ for $\nu \in V_N$, and λ_N is the highest weight of V_N. This proves (1-80). Note that if λ is on the boundary of the fundamental Weyl chamber, not all of these ν actually occur in the decomposition; nevertheless, the characteristic equation holds.

Chapter 2

Field theory on the q-deformed fuzzy sphere

In this section we elaborate the simplest case of Chapter 1, which are spherical branes on the group $SU(2)$. We will ignore here the “target” space $G = SU(2)$, and study the structure of the individual branes, which are q–deformed fuzzy spheres $\mathcal{S}^2_{q,N}$. Those are precisely the branes which were briefly exhibited in Section 1.7.2. This chapter is based[1] on the paper [36] which was written in collaboration with Harald Grosse and John Madore.

After reviewing the undeformed fuzzy sphere, we give a definition of $\mathcal{S}^2_{q,N}$ in Section 2.2 for both $q \in \mathbb{R}$ and $|q| = 1$. As an algebra, it is simply a finite–dimensional matrix algebra, equipped with additional structure such as an action of $U_q(su(2))$, a covariant differential calculus, a star structure, and an integral. For $q \in \mathbb{R}$, this is precisely the “discrete” series of Podleś spheres [67]. For $|q| = 1$, the algebra (1-105) is reproduced, with $N \equiv n$. This case, which is relevant to string theory as discussed before, has apparently not been studied in detail before. In Section 2.3, we develop the non–commutative differential geometry on $\mathcal{S}^2_{q,N}$, using an approach which is suitable for both $q \in \mathbb{R}$ and $|q| = 1$. The differential calculus turns out to be elaborate, but quite satisfactory. We are able to show, in particular, that in both cases there exists a 3–dimensional exterior differential calculus with real structure and a Hodge star, and we develop a frame formalism [68, 69, 70]. This allows us to write Lagrangians for field theories on $\mathcal{S}^2_{q,N}$. In particular, the fact that the tangential space is 3–dimensional unlike in the classical case is very interesting physically, reflecting the fact that the D–branes are embedded in a higher-dimensional space.

Using these tools, we study in Section 2.4.1 actions for scalar fields and abelian gauge fields on $\mathcal{S}^2_{q,N}$. The latter case is particularly interesting, since it turns out that certain actions for gauge theories arise in a very natural way in terms of polynomials of one–forms. In particular, the kinetic terms arise automatically due to the noncommutativity of the space. Moreover, because the calculus is 3–dimensional, the gauge field consists of a usual (abelian) gauge field plus a (pseudo) scalar in the classical limit. This is similar to a Kaluza–Klein reduction. One naturally obtains

[1] I adapted the conventions to those of chapter 1. In particular, the coproduct is reversed from [36]

analogs of Yang–Mills and Chern–Simons actions, again because the calculus is 3–dimensional. In a certain limit where $q = 1$, such actions were shown to arise from open strings ending on D–branes in the $SU(2)$ WZW model [30]. The gauge theory actions for $q \neq 1$ suggest a new version of gauge invariance, where the gauge "group" is a quotient $U_q(su(2))/I$, which can be identified with the space of functions on the deformed fuzzy sphere. This is discussed in Section 2.4.2.

In this chapter, we shall only consider the first–quantized situation; the second quantization is discussed in Chapter 5. The latter turns out to be necessary for implementing the symmetry $U_q(su(2))$ on the space of fields in a fully satisfactory way.

We should perhaps add a general remark on the type of noncommutative spaces considered here. It is customary in the literature on noncommutative geometry to define the dimension of a noncommutative space in terms of so-called Chern-Connes characters [2], which are related to the K theory of the underlying function algebras. In this framework, all fuzzy spheres are "zero-dimensional", being finite matrix algebras. However it will become clear in the later sections that this does not correspond to the physical concept of dimension in physics. We will see explicitly that the field theories defined on these spaces do behave as certain "regularized" field theories on ordinary spheres. The reason is the harmonic analysis: if decomposed under the appropriate symmetry algebra, the space or algebra of functions is for "low energies" (=for small representations) exactly the same as classically. It is the number of degrees of freedom below a given cutoff Λ (more precisely its scaling with Λ) which determines the physical dimension of a space, and enters the important quantities such as entropy, spectral action etc. We will therefore not worry about these K-theoretic issues any further here.

2.1 The undeformed fuzzy sphere

We give a quick review of the "standard" fuzzy sphere [29, 61, 71]. Much information about the standard unit sphere S^2 in $\mathbb{R}^3$ is encoded in the infinite dimensional algebra of polynomials generated by $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{R}^3$ with the defining relations

$$[\tilde{x}_i, \tilde{x}_j] = 0, \quad \sum_{i=1}^{3} \tilde{x}_i^2 = r^2 \tag{2-1}$$

The algebra of functions on the fuzzy sphere is defined as the finite algebra $\mathcal{S}_N^2$ generated by $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$, with relations

$$[\hat{x}_i, \hat{x}_j] = i\lambda_N \eta_{ijk} \hat{x}_k \tag{2-2}$$

$$\sum_{i=1}^{3} \hat{x}_i^2 = r^2 \tag{2-3}$$

The real parameter $\lambda_N > 0$ characterizes the non-commutativity, and has the dimension of a length. The radius r is quantized in units of λ_N by

$$\frac{r}{\lambda_N} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}, \quad N = 1, 2, \cdots \tag{2-4}$$

This quantization can be easily understood. Indeed (2-2) is simply the Lie algebra $su(2)$, whose irreducible representation are labeled by the spin $N/2$. The Casimir of the spin-$N/2$ representation is quantized, and related to r^2 by (4-1). The fuzzy sphere S^2_N is thus characterized by its radius r and the "noncommutativity parameters" N or λ_N. The integral of a function $F \in S^2_N$ over the fuzzy sphere is given by

$$\int F = \frac{4\pi R^2}{N+1}\mathrm{tr}[F(x)], \tag{2-5}$$

It agrees with the usual integral on S^2 in the large N limit. Invariance of the integral under the rotations $SU(2)$ amounts to invariance of the trace under adjoint action.

The algebra of functions is most conveniently realized using the Wigner-Jordan realization of the generators $\hat{x}_i$, $i = 1, 2, 3$, in terms of two pairs of annihilation and creation operators $A_\alpha, A^{+\alpha}, \alpha = \pm\frac{1}{2}$, which satisfy

$$[A_\alpha, A_\beta] = [A^{+\alpha}, A^{+\beta}] = 0 \;, \;\; [A_\alpha, A^{+\beta}] = \delta^\beta_\alpha \;, \tag{2-6}$$

and act on the Fock space $\mathcal{F}$ spanned by the vectors

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}}(A^{+\frac{1}{2}})^{n_1}(A^{+-\frac{1}{2}})^{n_2}|0\rangle \;. \tag{2-7}$$

Here $|0\rangle$ is the vacuum defined by $A_i|0\rangle = 0$. The operators $\hat{x}_i$ take the form

$$\hat{x}_i = \frac{\lambda_N}{\sqrt{2}}A^{+\alpha'}\varepsilon_{\alpha'\alpha}\sigma_i^{\alpha\beta}A_\beta. \tag{2-8}$$

Here $\varepsilon_{\alpha\alpha'}$ is the antisymmetric tensor (spinor metric), and $\sigma_i^{\alpha\beta}$ are the Clebsch-Gordan coefficients, that is rescaled Pauli–matrices. The number operator is given by $\hat{N} = \sum_\alpha A^{+\alpha}A_\alpha$. When restricted to the $(N+1)$-dimensional subspace

$$\mathcal{F}_N = \{\sum A^{+\alpha_1} ... \; A^{+\alpha_N}|0\rangle \; (N \text{ creation operators})\}. \tag{2-9}$$

it yields for any given $N = 0, 1, 2, ...$ the irreducible unitary representation in which the parameters λ_N and r are related as

$$\frac{r}{\lambda_N} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)} \;. \tag{2-10}$$

The algebra $\mathcal{S}^2_N$ generated by the $\hat{x}_i$ is clearly the simple matrix algebra $Mat(N+1)$. Under the adjoint action of $SU(2)$, it decomposes into the direct sum $(1) \oplus (3) \oplus (5) \oplus ... \oplus (2N+1)$ of irreducible representations of $SO(3)$ [61, 71].

2.2 The q-deformed fuzzy sphere

2.2.1 The q–deformed algebra of functions

The fuzzy sphere $\mathcal{S}^2_N$ is invariant under the action of $SU(2)$, or equivalently under the action of $U(su(2))$. We shall define finite algebras $\mathcal{S}^2_{q,N}$ generated by x_i for $i = 1, 0, -1$, which have completely analogous properties to those of $\mathcal{S}^2_N$, but which are covariant under the quantized universal enveloping algebra $U_q(su(2))$. This will be done for both $q \in \mathbb{R}$ and q a phase, including the appropriate reality structure. In the first case, the $\mathcal{S}^2_{q,N}$ will turn out to be the "discrete series" of Podleś' quantum spheres [67]. Here we will study them more closely from the above point of view. However, we also allow q to be a root of unity, with certain restrictions. In a twisted form, this case does appear naturally on D–branes in the $SU(2)$ WZW model, as was shown in [8].

In order to make the analogy to the undeformed case obvious, we perform a q–deformed Jordan–Wigner construction, which is covariant under $U_q(su(2))$. To fix the notation, we recall the basic relations of $U_q(su(2))$

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}} = [H]_q \tag{2-11}$$

where the q–numbers are defined as $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The action of $U_q(su(2))$ on a tensor product of representations is encoded in the coproduct

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(X^\pm) = X^\pm \otimes q^{H/2} + q^{-H/2} \otimes X^\pm. \tag{2-12}$$

The antipode and the counit are given by

$$S(H) = -H, \quad S(X^\pm) = -q^{\pm 1} X^\pm, \qquad \varepsilon(H) = \varepsilon(X^\pm) = 0. \tag{2-13}$$

The star structure is related to the Cartan–Weyl involution $\theta(X^\pm) = X^\mp,\ \theta(H) = H$, and will be discussed below. All symbols will now be understood to carry a label "q", which we shall omit.

Consider q–deformed creation and anihilation operators $A_\alpha, A^{+\alpha}$ for $\alpha = \pm\frac{1}{2}$, which satisfy the relations (cp. [72, 73])

$$\begin{aligned} A^{+\alpha} A_\beta &= \delta^\alpha_\beta + q \hat{R}^{\alpha\gamma}_{\beta\delta} A_\gamma A^{+\delta} \\ (P^-)^{\alpha\beta}_{\gamma\delta} A_\alpha A_\beta &= 0 \\ (P^-)^{\alpha\beta}_{\gamma\delta} A^{+\delta} A^{+\gamma} &= 0 \end{aligned} \tag{2-14}$$

where $\hat{R}^{\alpha\gamma}_{\beta\delta} = q(P^+)^{\alpha\gamma}_{\beta\delta} - q^{-1}(P^-)^{\alpha\gamma}_{\beta\delta}$ is the decomposition of the $\hat{R}$ –matrix of $U_q(su(2))$ into the projection operators on the symmetric and antisymmetric part. They can be written as

$$\begin{aligned} (P^-)^{\alpha\beta}_{\gamma\delta} &= \frac{1}{-[2]_q} \varepsilon^{\alpha\beta} \varepsilon_{\gamma\delta}, \\ (P^+)^{\alpha\beta}_{\gamma\delta} &= \sigma^{\alpha\beta}_i \sigma^i_{\gamma\delta} \end{aligned} \tag{2-15}$$

Here $\varepsilon_{\alpha\beta}$ is the q–deformed invariant antisymmetric tensor (see Section 1.4.2), and $\sigma^i_{\alpha\beta}$ are the q–deformed Clebsch–Gordan coefficients; they are given explicitly in Section 2.6.1. The factor $-[2]_q^{-1}$ arises from the relation $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = -[2]_q$. The above relations are covariant under $U_q(su(2))$, and define a left $U_q(su(2))$–module algebra. We shall denote the action on the generators with lower indices by

$$u \triangleright A_\alpha = A_\beta \; \pi^\beta_\alpha(u), \tag{2-16}$$

so that $\pi^\alpha_\beta(uv) = \pi^\alpha_\gamma(u)\pi^\gamma_\beta(v)$ for $u, v \in U_q(su(2))$. The generators with upper indices transform in the contragredient representation, which means that

$$A^+_\alpha := \varepsilon_{\alpha\beta} A^{+\beta} \tag{2-17}$$

transforms in the same way under $U_q(su(2))$ as A_α.

We consider again the corresponding Fock space $\mathcal{F}$ generated by the $A^{+\alpha}$ acting on the vacuum $|0\rangle$, and its sectors

$$\mathcal{F}_N = \{\sum A^{+\alpha_1} ... \; A^{+\alpha_N}|0\rangle \; (N \text{ creation operators})\}. \tag{2-18}$$

It is well–known that these subspaces $\mathcal{F}_N$ are $N+1$–dimensional, as they are when $q = 1$, and it follows that they form irreducible representations of $U_q(su(2))$ (at root of unity, this will be true due to the restriction (2-38) we shall impose). This will be indicated by writing $\mathcal{F}_N = (N+1)$, and the decomposition of $\mathcal{F}$ into irreducible representations is

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus ... = (1) \oplus (2) \oplus (3) \oplus ... \tag{2-19}$$

Now we define

$$\hat{Z}_i = A^{+\alpha'} \varepsilon_{\alpha\alpha'} \sigma^{\alpha\beta}_i A_\beta \tag{2-20}$$

and

$$\hat{N} = \sum_\alpha A^{+\alpha'} \varepsilon_{\alpha\alpha'} \varepsilon^{\alpha\beta} A_\beta. \tag{2-21}$$

After some calculations, these operators can be shown to satisfy the relations

$$\varepsilon^{ij}_k \hat{Z}_i \hat{Z}_j = \frac{q^{-1}}{\sqrt{[2]_q}} (q^{-1}[2]_q - \lambda \hat{N}) \hat{Z}_k \tag{2-22}$$

$$\hat{Z}^2 := g^{ij} \hat{Z}_i \hat{Z}_j = q^{-2} \frac{[2]_q + \hat{N}}{[2]_q} \hat{N} \tag{2-23}$$

Here $\lambda = (q - q^{-1})$, g^{ij} is the q–deformed invariant tensor for spin 1 representations, and ε^{ij}_k is the corresponding q–deformed Clebsch–Gordan coefficient; they are given in Section 2.6.1. Moreover, one can verify that

$$\begin{aligned} \hat{N} A^{+\alpha} &= q^{-3} A^{+\alpha} + q^{-2} A^{+\alpha} \hat{N}, \\ \hat{N} A_\alpha &= -q^{-1} A_\alpha + q^2 A_\alpha \hat{N}, \end{aligned} \tag{2-24}$$

which implies that

$$[\hat{N}, \hat{Z}_i] = 0.$$

On the subspace $\mathcal{F}_N$, the "number" operator $\hat{N}$ takes the value

$$\hat{N}\mathcal{F}_N = q^{-N-2}[N]_q\mathcal{F}_N \tag{2-25}$$

It is convenient to introduce also an undeformed number operator $\hat{n}$ which has eigenvalues

$$\hat{n}\mathcal{F}_N = N\mathcal{F}_N,$$

in particular $\hat{n}A_\alpha = A_\alpha(\hat{n}-1)$.

On the subspaces $\mathcal{F}_N$, the relations (2-22) become

$$\begin{aligned} \varepsilon_k^{ij} x_i x_j &= \Lambda_N\, x_k, & (2\text{-}26)\\ x\cdot x &:= g^{ij}x_i x_j = r^2. & (2\text{-}27) \end{aligned}$$

Here the variables have been rescaled to x_i with

$$x_i = r\,\frac{q^{\hat{n}+2}}{\sqrt{[2]_q C_N}}\,\hat{Z}_i.$$

The r is a real number, and we have defined

$$\begin{aligned} C_N &= \frac{[N]_q[N+2]_q}{[2]_q^2},\\ \Lambda_N &= r\,\frac{[2]_{q^{N+1}}}{\sqrt{[N]_q[N+2]_q}}. & (2\text{-}28) \end{aligned}$$

Using a completeness relation (see Section 2.6.1), (2-26) can equivalently be written as

$$(P^-)^{ij}_{kl}x_i x_j = \frac{1}{[2]_{q^2}}\,\Lambda_N\varepsilon^n_{kl}\,x_n. \tag{2-29}$$

There is no i in the commutation relations, because we use a weight basis instead of Cartesian coordinates. One can check that these relations precisely reproduce the "discrete" series of Podleś' quantum spheres (after another rescaling), see [67], Proposition 4.II. Hence we define $\mathcal{S}^2_{q,N}$ to be the algebra generated by the variables x_i acting on $\mathcal{F}_N$. Equipped with a suitable star structure and a differential structure, this will be the q–deformed fuzzy sphere.

It is easy to see that the algebra $\mathcal{S}^2_{q,N}$ is simply the full matrix algebra $Mat(N+1)$, i.e. it is the same algebra as $\mathcal{S}^2_N$ for $q=1$. This is because $\mathcal{F}_N$ is an irreducible representation of $U_q(su(2))$. To see it, we use complete reducibility [74] of the space of polynomials in x_i of degree $\leq k$ to conclude that it decomposes into the direct sum of irreducible representations $(1)\oplus(3)\oplus(5)\oplus...\oplus(2k+1)$. Counting dimensions and noting that $x_1^N \neq 0 \in (2N+1)$, it follows that $\dim(\mathcal{S}^2_{q,N}) = (N+1)^2 = \dim Mat(N+1)$, and hence

$$\boxed{\mathcal{S}^2_{q,N} = (1)\oplus(3)\oplus(5)\oplus...\oplus(2N+1)} \tag{2-30}$$

(as in (1-87)). This is true even if q is a root of unity provided (2-38) below holds, a relation which will be necessary for other reasons as well. This is the decomposition of the functions on the q–deformed fuzzy sphere into q–spherical harmonics, and it is automatically truncated. Note however that not all information about a (quantum) space is encoded in its algebra of functions; in addition, one must specify for example a differential calculus and symmetries. For example, the action of $U_q(su(2))$ on $\mathcal{S}^2_{q,N}$ is different from the action of $U(su(2))$ on $\mathcal{S}^2_N$.

The covariance of $\mathcal{S}^2_{q,N}$ under $U_q(su(2))$ can also be stated in terms of the quantum adjoint action. It is convenient to consider the cross–product algebra $U_q(su(2)) \ltimes \mathcal{S}^2_{q,N}$, which as a vector space is equal to $U_q(su(2)) \otimes \mathcal{S}^2_{q,N}$, equipped with an algebra structure defined by

$$ux = (u_{(1)} \triangleright x) u_{(2)}. \tag{2-31}$$

Here the $\triangleright$ denotes the action of $u \in U_q(su(2))$ on $x \in \mathcal{S}^2_{q,N}$. Conversely, the action of $U_q(su(2))$ on $\mathcal{S}^2_{q,N}$ can be written as $u \triangleright x = u_{(1)} x S u_{(2)}$. The relations (2-31) of $U_q(su(2)) \ltimes \mathcal{S}^2_{q,N}$ are automatically realized on the representation $\mathcal{F}_N$.

Since both algebras $\mathcal{S}^2_{q,N}$ and $U_q(su(2))$ act on $\mathcal{F}_N$ and generate the full matrix algebra $Mat(N+1)$, it must be possible to express the generators of $U_q(su(2))$ in terms of the Z_i. The explicit relation can be obtained by comparing the relations (2-26) with (2-31). One finds

$$\begin{aligned} X^+ q^{-H/2} &= q^{N+3}\, \hat{Z}_1, \\ X^- q^{-H/2} &= -q^{N+1}\, \hat{Z}_{-1}, \\ q^{-H} &= \frac{[2]_{q^{N+1}}}{[2]_q} + \frac{q^{N+2}(q-q^{-1})}{\sqrt{[2]_q}}\, \hat{Z}_0, \end{aligned} \tag{2-32}$$

if acting on $\mathcal{F}_N$. In fact, this defines an algebra map

$$j: \quad U_q(su(2)) \to \mathcal{S}^2_{q,N} \tag{2-33}$$

which satisfies

$$\boxed{j(u_{(1)}) x j(S u_{(2)}) = u \triangleright x} \tag{2-34}$$

for $x \in \mathcal{S}^2_{q,N}$ and $u \in U_q(su(2))$. This is analogous to results in [75, 76]. We shall often omit j from now on. In particular, $\mathcal{S}^2_{q,N}$ is the quotient of the algebra $U_q(su(2))$ by the relation (2-27). The relations (2-34) and those of $U_q(su(2))$ can be verified explicitly using (2-22). Moreover, one can verify that it is represented correctly on $\mathcal{F}_N$ by observing that $X^+ (A^+_{1/2})^N |0\rangle = 0$, which means that $(A^+_{1/2})^N |0\rangle$ is the highest–weight vector of $\mathcal{F}_N$.

2.2.2 Reality structure for $q \in \mathbb{R}$

In order to define a real quantum space, we must also construct a star structure, which is an involutive anti–linear anti–algebra map. For real q, the algebra (2-14) is consistent with the following star structure

$$\begin{aligned} (A_\alpha)^* &= A^{+\alpha} \\ (A^{+\alpha})^* &= A_\alpha \end{aligned} \tag{2-35}$$

This can be verified using the standard compatibility relations of the $\hat{R}$–matrix with the invariant tensor [24]. On the generators x_i, it implies the relation

$$x_i^* = g^{ij} x_j, \tag{2-36}$$

as well as the equality

$$\hat{N}^* = \hat{N}.$$

The algebras $\mathcal{S}^2_{q,N}$ are now precisely Podleś' "discrete" C^* algebras $\tilde{S}^2_{q,c(N+1)}$. Using (2-32), this is equivalent to

$$H^* = H, \ (X^{\pm})^* = X^{\mp}, \tag{2-37}$$

which is the star–structure for the compact form $U_q(su(2))$. It is well–known that there is a unique Hilbert space structure on the subspaces $\mathcal{F}_N$ such that they are unitary irreducible representations of $U_q(su(2))$. Then the above star is simply the operator adjoint.

2.2.3 Reality structure for q a phase

When q is a phase, finding the correct star structure is not quite so easy. The difference with the case $q \in \mathbb{R}$ is that $\Delta(u^*) = (* \otimes *)\Delta'(u)$ for $|q| = 1$ and $u \in U_q(su(2))$, where Δ' denotes the flipped coproduct. We shall define a star only on the algebra $\mathcal{S}^2_{q,N}$ generated by the x_i, and not on the full algebra generated by A_α and A^+_α.

There appears to be an obvious choice at first sight, namely $x_i^* = x_i$, which is indeed consistent with (2-26). However, it is the wrong choice for our purpose, because it induces the noncompact star structure $U_q(sl(2,\mathbb{R}))$.

Instead, we define a star–structure on $\mathcal{S}^2_{q,N}$ as follows. The algebra $U_q(su(2))$ acts on the space $\mathcal{S}^2_{q,N}$, which generically decomposes as $(1) \oplus (3) \oplus ... \oplus (2N+1)$. This decomposition should be a direct sum of *unitary* representations of the compact form of $U_q(su(2))$, which means that the star structure on $U_q(su(2))$ should be (2-37), as it is for real q. There is a slight complication, because not all finite–dimensional irreducible representations are unitary if q is a phase [77]. However, all representations with dimension $\leq 2N+1$ are unitary provided q has the form

$$q = e^{i\pi\varphi}, \quad \text{with} \ \ \varphi < \frac{1}{2N}. \tag{2-38}$$

This will be assumed from now on.

As was pointed out before, we can consider the algebra $\mathcal{S}^2_{q,N}$ as a quotient of $U_q(su(2))$ via (2-32). It acts on $\mathcal{F}_N$, which is an irreducible representation of $U_q(su(2))$, and hence has a natural Hilbert space structure. We define the star on the operator algebra $\mathcal{S}^2_{q,N}$ by the adjoint (that is by the matrix adjoint in an orthonormal basis), hence by the star (2-37) using the identification (2-32).

There is a very convenient way to write down this star structure on the generators x_i, similar as in [78]. It involves an element ω of an extension of $U_q(su(2))$ introduced by [64] and [79],

which implements the Weyl reflection on irreducible representations. The essential properties are

$$\begin{aligned} \Delta(\omega) &= \mathcal{R}^{-1}\omega\otimes\omega, && (2\text{-}39)\\ \omega u\omega^{-1} &= \theta S^{-1}(u), && (2\text{-}40)\\ \omega^2 &= v\epsilon, && (2\text{-}41) \end{aligned}$$

where v and ϵ are central elements in $\in U_q(su(2))$ which take the values $q^{-N(N+2)/2}$ resp. $(-1)^N$ on $\mathcal{F}_N$. Here $\mathcal{R} = \mathcal{R}_1\otimes\mathcal{R}_2 \in U_q(su(2))\otimes U_q(su(2))$ is the universal R element. In a unitary representation of $U_q(su(2))$, the matrix representing ω in an orthonormal basis is given the invariant tensor in a certain normalization, $\pi^i_j(\omega) = -q^{-N(N+2)/4}g^{ij}$, and $\omega^* = \omega^{-1}$. This is discussed in detail in [78]. From now on, we denote with ω the element in $\mathcal{S}^2_{q,N}$ which represents this element on $\mathcal{F}_N$.

We claim that the star structure on $\mathcal{S}^2_{q,N}$ as explained above is given by the following formula:

$$x_i^* = -\omega x_i\omega^{-1} = x_j L^{-j}_{\ \ k} q^{-2}g^{ki}, \tag{2-42}$$

where

$$L^{-i}_{\ \ j} = \pi^i_j(\mathcal{R}_1^{-1})\mathcal{R}_2^{-1} \tag{2-43}$$

as usual [24]; a priori, $L^{-i}_{\ \ i} \in U_q(su(2))$, but it is understood here as an element of $\mathcal{S}^2_{q,N}$ via (2-32). One can easily verify using $(\varepsilon^{ij}_k)^* = -\varepsilon^{ji}_k$ (for $|q| = 1$) that (2-42) is consistent with the relations (2-26) and (2-27). In the limit $q\to 1$, $L^{-i}_{\ \ j}\to\delta^i_j$, therefore (2-42) agrees with (2-36) in the classical limit. Hence we define the q–deformed fuzzy sphere for q a phase to be the algebra $\mathcal{S}^2_{q,N}$ equipped with the star–structure (2-42).

To show that (2-42) is correct in the sense explained above, it is enough to verify that it induces the star structure (2-37) on $U_q(su(2))$, since both $U_q(su(2))$ and $\mathcal{S}^2_{q,N}$ generate the same algebra $Mat(N+1)$. This can easily be seen using (2-40) and (2-32). A somewhat related conjugation has been proposed in [78, 80] using the universal element $\mathcal{R}$.

2.2.4 Invariant integral

The integral on $\mathcal{S}^2_{q,N}$ is defined to be the unique functional on $\mathcal{S}^2_{q,N}$ which is invariant under the (quantum adjoint) action of $U_q(su(2))$. It is given by the projection on the trivial sector in the decomposition (2-30). We claim that it can be written explicitly using the quantum trace:

$$\int\limits_{\mathcal{S}^2_{q,N}} f(x_i) := 4\pi r^2\frac{1}{[N+1]_q}\mathrm{Tr}_q(f(x_i)) = 4\pi r^2\frac{1}{[N+1]_q}\mathrm{Tr}(f(x_i)\; q^H) \tag{2-44}$$

for $f(x_i)\in\mathcal{S}^2_{q,N}$, where the trace is taken on $\mathcal{F}_N$. Using $S^{-2}(u) = q^{-H}uq^H$ for $u\in U_q(su(2))$, it follows that

$$\int\limits_{\mathcal{S}^2_{q,N}} fg = \int\limits_{\mathcal{S}^2_{q,N}} S^{-2}(g)f. \tag{2-45}$$

This means that it is indeed invariant under the quantum adjoint action[2],

$$\begin{aligned}\int\limits_{\mathcal{S}^2_{q,N}} u \triangleright f(x_i) &= \int\limits_{\mathcal{S}^2_{q,N}} u_1 f(x_i) S(u_2) \\ &= \int\limits_{\mathcal{S}^2_{q,N}} S^{-1}(u_2) u_1 f(x_i) = \varepsilon(u) \int\limits_{\mathcal{S}^2_{q,N}} f(x_i),\end{aligned} \tag{2-46}$$

using the identification (2-32). The normalization constant is obtained from

$$\mathrm{Tr}_q(1) = \mathrm{Tr}(q^H) = q^N + q^{N-2} + ... + q^{-N} = [N+1]_q$$

on $\mathcal{F}_N$, so that $\int\limits_{\mathcal{S}^2_{q,N}} 1 = 4\pi r^2$.

Lemma 2.2.1. *Let $f \in \mathcal{S}^2_{q,N}$. Then*

$$\Big(\int\limits_{\mathcal{S}^2_{q,N}} f\Big)^* = \int\limits_{\mathcal{S}^2_{q,N}} f^* \tag{2-47}$$

for real q, and

$$\Big(\int\limits_{\mathcal{S}^2_{q,N}} f\Big)^* = \int\limits_{\mathcal{S}^2_{q,N}} f^* q^{-2H} \tag{2-48}$$

for q a phase, with the appropriate star structure (2-36) respectively (2-42). In (2-48), we use (2-32).

Proof. Assume first that q is real, and consider the functional

$$I_{q,N}(f) := \mathrm{Tr}(f^* q^H)^*$$

for $f \in \mathcal{S}^2_{q,N}$. Then

$$\begin{aligned}I_{q,N}(u \triangleright f) &= \mathrm{Tr}((u_1 f S(u_2))^* \, q^H)^* \\ &= \mathrm{Tr}(S^{-1}((u^*)_2) f^* (u^*)_1 \, q^H)^* = \mathrm{Tr}(f^*(u^*)_1 S((u^*)_2) q^H)^* \\ &= \varepsilon(u) \, I_{q,N}(f),\end{aligned} \tag{2-49}$$

where $(S(u))^* = S^{-1}(u^*)$ and $(* \otimes *)\Delta(u) = \Delta(u^*)$ was used. Hence $I_{q,N}(f)$ is invariant as well, and (2-47) follows using uniqueness of the integral (up to normalization). For $|q| = 1$, we define

$$\tilde{I}_{q,N}(f) := \mathrm{Tr}(f^* q^{-H})^*$$

with the star structure (2-42). Using $(S(u))^* = S(u^*)$ and $(* \otimes *)\Delta(u) = \Delta'(u^*)$, an analogous calculation shows that $\tilde{I}_{q,N}$ is invariant under the action of $U_q(su(2))$, which again implies (2-48). □

[2]Note the difference to (1-65): there, the trace is over the *indices* of the coordinates M^i_j, not over the representation. This leads to q^{-H} rather than q^H.

For $|q|=1$, the integral is neither real nor positive, hence it cannot be used for a GNS construction. Nevertheless, it is clearly the appropriate functional to define an action for field theory, since it is invariant under $U_q(su(2))$. To find a way out, we introduce an auxiliary antilinear algebra–map on $\mathcal{S}^2_{q,N}$ by

$$\overline{f} = S^{-1}(f^*) \tag{2-50}$$

where S is the antipode on $U_q(su(2))$, using (2-32). Note that S preserves the relation (2-27), hence it is well–defined on $\mathcal{S}^2_{q,N}$. This is not a star structure, since

$$\overline{\overline{f}} = S^{-2} f$$

for $|q|=1$. Using (2-32), one finds in particular

$$\overline{x_i} = -g^{ij} x_j. \tag{2-51}$$

This is clearly consistent with the relations (2-26) and (2-27). We claim that (2-48) can now be stated as

$$\Big(\int\limits_{\mathcal{S}^2_{q,N}} f\Big)^* = \int\limits_{\mathcal{S}^2_{q,N}} \overline{f} \qquad \text{for } \; |q|=1. \tag{2-52}$$

To see this, observe first that

$$\text{Tr}(S(f)) = \text{Tr}(f), \tag{2-53}$$

which follows either from the fact that $\hat{I}_{q,N}(f) := \text{Tr}(S^{-1}(f)q^{-H}) = \text{Tr}(S^{-1}(q^H f))$ is yet another invariant functional, or using $\omega f \omega^{-1} = \theta S^{-1}(f)$ together with the observation that the matrix representations of $X^\pm$ in a unitary representation are real. This implies

$$\text{Tr}_q(f^* q^{-2H}) = \text{Tr}(f^* q^{-H}) = \text{Tr}(S(q^H S^{-1}(f^*))) = \text{Tr}(q^H S^{-1}(f^*)) = \text{Tr}_q(\overline{f}), \tag{2-54}$$

and (2-52) follows. Now we can write down a positive inner product on $\mathcal{S}^2_{q,N}$:

Lemma 2.2.2. *The sesquilinear forms*

$$(f,g) := \int\limits_{\mathcal{S}^2_{q,N}} f^* g \quad \textit{for} \;\; q \in \mathbb{R} \tag{2-55}$$

and

$$(f,g) := \int\limits_{\mathcal{S}^2_{q,N}} \overline{f} g, \quad \textit{for} \;\; |q| = 1 \tag{2-56}$$

are hermitian, that is $(f,g)^* = (g,f)$*, and satisfy*

$$(f, u \triangleright g) = (u^* \triangleright f, g) \tag{2-57}$$

for both $q \in \mathbb{R}$ *and* $|q|=1$*. They are positive definite provided (2-38) holds for* $|q|=1$*, and define a Hilbert space structure on* $\mathcal{S}^2_{q,N}$*.*

Proof. For $q \in \mathbb{R}$, we have

$$\begin{aligned}(f, u \triangleright g) &= \int\limits_{\mathcal{S}^2_{q,N}} f^* u_1 g S u_2 = \int\limits_{\mathcal{S}^2_{q,N}} S^{-1}(u_2) f^* u_1 g = \int\limits_{\mathcal{S}^2_{q,N}} (S((u^*)_2)^* f^* ((u^*)_1)^* g \\ &= \int\limits_{\mathcal{S}^2_{q,N}} ((u^*)_1 f S(u^*)_2)^* \, g = (u^* \triangleright f, g), \end{aligned} \tag{2-58}$$

and hermiticity is immediate. For $|q| = 1$, consider

$$\begin{aligned}(f, u \triangleright g) &= \int\limits_{\mathcal{S}^2_{q,N}} \overline{f} u_1 g S u_2 = \int\limits_{\mathcal{S}^2_{q,N}} S^{-1}(u_2) S^{-1}(f^*) u_1 g = \int\limits_{\mathcal{S}^2_{q,N}} S^{-1} \left((u^*)_1 f S(u^*)_2\right)^* g \\ &= (u^* \triangleright f, g). \end{aligned} \tag{2-59}$$

Hermiticity follows using (2-52):

$$(f, g)^* = \int\limits_{\mathcal{S}^2_{q,N}} \overline{\overline{f}} \overline{g} = \int\limits_{\mathcal{S}^2_{q,N}} S^{-2}(f) \overline{g} = \int\limits_{\mathcal{S}^2_{q,N}} \overline{g} f = (g, f).$$

Using the assumption (2-38) for $|q| = 1$, it is not difficult to see that they are also positive–definite. □

2.3 Differential Calculus

In order to write down Lagrangians, it is convenient to use the notion of an (exterior) differential calculus [81, 2]. A covariant differential calculus over $\mathcal{S}^2_{q,N}$ is a graded bimodule $\Omega^*_{q,N} = \oplus_n \Omega^n_{q,N}$ over $\mathcal{S}^2_{q,N}$ which is a $U_q(su(2))$–module algebra, together with an exterior derivative d which satisfies $d^2 = 0$ and the graded Leibniz rule. We define the dimension of a calculus to be the rank of $\Omega^1_{q,N}$ as a free right $\mathcal{S}^2_{q,N}$–module.

2.3.1 First–order differential forms

Differential calculi for the Podleś sphere have been studied before [82, 83]. It turns out that 2–dimensional calculi do not exist for the cases we are interested in; however there exists a unique 3–dimensional module of 1–forms. As opposed to the classical case, it contains an additional "radial" one–form. This will lead to an additional scalar field, which will be discussed later.

By definition, it must be possible to write any term $x_i dx_j$ in the form $\sum_k dx_k f_k(x)$. Unfortunately the structure of the module of 1–forms turn out to be not quadratic, rather the $f_k(x)$ are polynomials of order up to 3. In order to make it more easily tractable and to find suitable reality structures, we will construct this calculus using a different basis. First, we will define the

bimodule of 1–forms $\Omega^1_{q,N}$ over $\mathcal{S}^2_{q,N}$ which is covariant under $U_q(su(2))$, such that $\{dx_i\}_i$ is a free right $\mathcal{S}^2_{q,N}$–module basis, together with a map $d: \mathcal{S}^2_{q,N} \to \Omega^1_{q,N}$ which satisfies the Leibnitz rule. Higher–order differential forms will be discussed below.

Consider a basis of one–forms ξ_i for $i = -1, 0, 1$ with the covariant commutation relations[3]

$$x_i \xi_j = \hat{R}^{kl}_{ij} \xi_k x_l, \tag{2-60}$$

using the $(3) \otimes (3)$ $\hat{R}$–matrix of $U_q(su(2))$. It has the projector decomposition

$$\hat{R}^{kl}_{ij} = q^2 (P^+)^{kl}_{ij} - q^{-2} (P^-)^{kl}_{ij} + q^{-4} (P^0)^{kl}_{ij}, \tag{2-61}$$

where $(P^0)^{kl}_{ij} = \frac{1}{[3]_q} g^{kl} g_{ij}$, and $(P^-)^{kl}_{ij} = \sum_n \frac{1}{[2]_{q^2}} \varepsilon^{kl}_n \varepsilon^n_{ij}$. The relations (2-60) are consistent with (2-27) and (2-26), using the braiding relations [66, 84]

$$\begin{aligned} \hat{R}^{kl}_{ij} \hat{R}^{rs}_{lu} \varepsilon^{ju}_n &= \varepsilon^{kr}_t \hat{R}^{ts}_{in}, && (2\text{-}62) \\ \hat{R}^{kl}_{ij} \hat{R}^{rs}_{lu} g^{ju} &= g^{kr} \delta^s_i && (2\text{-}63) \end{aligned}$$

and the quantum Yang–Baxter equation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$, in shorthand–notation [24]. We define $\Omega^1_{q,N}$ to be the free right module over $\mathcal{S}^2_{q,N}$ generated by the ξ_i. It is clearly a bimodule over $\mathcal{S}^2_{q,N}$. To define the exterior derivative, consider

$$\Theta := x \cdot \xi = x_i \xi_j g^{ij}, \tag{2-64}$$

which is a singlet under $U_q(su(2))$. It turns out (see Section 2.6.2) that $[\Theta, x_i] \neq 0 \in \Omega^1_{q,N}$. Hence

$$\boxed{df := [\Theta, f(x)]} \tag{2-65}$$

defines a nontrivial derivation $d: \mathcal{S}^2_{q,N} \to \Omega^1_{q,N}$, which completes the definition of the calculus up to first order. In particular, it is shown in Section 2.6.2 that

$$dx_i = -\Lambda_N \varepsilon^{nk}_i x_n \xi_k + (q - q^{-1})(q x_i \Theta - r^2 q^{-1} \xi_i). \tag{2-66}$$

Since all terms are linearly independent, this is a 3–dimensional first–order differential calculus, and by the uniqueness it agrees with the 3–dimensional calculus in [82, 83]. In view of (2-66), it is not surprising that the commutation relations between the generators x_i and dx_i are very complicated [83]; will not write them down here. The meaning of the ξ–forms will become clearer in Section 2.3.4.

Using (2-150) and the relation $\xi \cdot x = q^4 x \cdot \xi$, one finds that

$$x \cdot dx = \left(-\Lambda_N^2 + ([2]_{q^2} - 2) r^2\right) \Theta.$$

On the other hand, this must be equal to $x_i \Theta x_j g^{ij} - r^2 \Theta$, which implies that

$$x_i \Theta x_j g^{ij} = \alpha r^2 \Theta$$

[3]note that this is not the same as $u\xi_i = u_{(1)} \triangleright \xi_i u_{(2)}$.

with

$$\alpha = [2]_{q^2} - 1 - \frac{\Lambda_N^2}{r^2} = 1 - \frac{1}{C_N}. \tag{2-67}$$

Combining this, it follows that

$$dx \cdot x = r^2 \, \frac{1}{C_N} \, \Theta = -x \cdot dx. \tag{2-68}$$

Moreover, using the identity (2-152) one finds

$$\varepsilon_i^{jk} x_j dx_k = (\alpha - q^2) r^2 \varepsilon_i^{jk} x_j \xi_k - \Lambda_N r^2 \xi_i + q^2 \Lambda_N x_i \Theta, \tag{2-69}$$

which together with (2-66) yields

$$\xi_i = \frac{q^2}{r^2} \, \Theta x_i + \frac{q^2 C_N \Lambda_N}{r^4} \varepsilon_i^{jk} x_j dx_k - q^2 (1 - q^2) \frac{C_N}{r^2} dx_i. \tag{2-70}$$

2.3.2 Higher–order differential forms

Podleś [82] has constructed an extension of the above 3–dimensional calculus including higher–order forms for a large class of quantum spheres. This class does not include ours, however, hence we will give a different construction based on ξ–variables, which will be suitable for q a phase as well.

Consider the algebra

$$\xi_i \xi_j = -q^2 \hat{R}^{kl}_{ij} \xi_k \xi_l \tag{2-71}$$

which is equivalent to $(P^+)^{ij}_{kl} \xi_i \xi_j = 0$, $(P^0)^{ij}_{kl} \xi_i \xi_j = 0$ where P^+ and P^0 are the projectors on the symmetric components of $(3) \otimes (3)$ as above; hence the product is totally (q–) antisymmetric. It is not hard to see (and well–known) that the dimension of the space of polynomials of order n in the ξ is $(3, 3, 1)$ for $n = (1, 2, 3)$, and zero for $n > 3$, as classically. We define $\Omega^n_{q,N}$ to be the free right $\mathcal{S}^2_{q,N}$–module with the polynomials of order n in ξ as basis; this is covariant under $U_q(su(2))$. Then $\Omega^n_{q,N}$ is in fact a (covariant) $\mathcal{S}^2_{q,N}$–bimodule, since the commutation relations (2-60) between x and ξ are consistent with (2-71), which follows from the quantum Yang–Baxter equation. There remains to construct the exterior derivative. To find it, we first note that (perhaps surprisingly) $\Theta^2 \neq 0$, rather

$$\Theta^2 = -\frac{q^{-2} \Lambda_N}{[2]_{q^2}} \varepsilon^{ijk} x_i \xi_j \xi_k. \tag{2-72}$$

The ε^{ijk} is defined in (2-153). By a straightforward but lengthy calculation which is sketched in Section 2.6.2, one can show that

$$dx_i dx_j g^{ij} + \frac{r^2}{C_N} \Theta^2 = 0.$$

We will show below that an extension of the calculus to higher–order forms exists; then this can be rewritten as

$$d\Theta - \Theta^2 = 0. \tag{2-73}$$

The fact that $\Theta^2 \neq 0$ makes the construction of the extension more complicated, since now $\alpha^{(n)} \to [\Theta, \alpha^{(n)}]_\pm$ does not define an exterior derivative. To remedy this, the following observation is useful: the map

$$\begin{aligned} *_H: \ \Omega^1_{q,N} &\to \Omega^2_{q,N}, \\ \xi_i &\mapsto -\frac{q^{-2}\Lambda_N}{[2]_{q^2}}\, \varepsilon_i^{jk}\xi_j\xi_k \end{aligned} \tag{2-74}$$

defines a left–and right $\mathcal{S}^2_{q,N}$–module map; in other words, the commutation relations between ξ_i and x_j are the same as between $*_H(\xi_i)$ and x_j. This follows from the braiding relation (2-62). This is in fact the natural analogue of the Hodge–star on 1–forms in our context, and will be discussed further below. Here we note the important identity

$$\alpha(*_H\beta) = (*_H\alpha)\beta \tag{2-75}$$

for any $\alpha, \beta \in \Omega^1_{q,N}$, which is proved in Section 2.6.2. Now (2-72) can be stated as

$$*_H(\Theta) = \Theta^2, \tag{2-76}$$

and applying $*_H$ to $df = [\Theta, f(Y)]$ one obtains

$$[\Theta^2, f(x)] = *_H df(x). \tag{2-77}$$

Now we define the map

$$\boxed{\begin{aligned} d: \ \Omega^1_{q,N} &\to \Omega^2_{q,N}, \\ \alpha &\mapsto [\Theta, \alpha]_+ - *_H(\alpha). \end{aligned}} \tag{2-78}$$

It is easy to see that this defines a graded derivation from $\Omega^1_{q,N}$ to $\Omega^2_{q,N}$, and the previous equation implies immediately that

$$(d \circ d) f = 0.$$

In particular,

$$d\xi_i = (1-q^2)\xi\Theta + \frac{q^{-2}\Lambda_N}{[2]_{q^2}}\varepsilon_i^{jk}\xi_j\xi_k. \tag{2-79}$$

To complete the differential calculus, we extend it to $\Omega^3_{q,N}$ by

$$\boxed{\begin{aligned} d: \ \Omega^2_{q,N} &\to \Omega^3_{q,N}, \\ \alpha^{(2)} &\mapsto [\Theta, \alpha^{(2)}]. \end{aligned}} \tag{2-80}$$

As is shown in Section 2.6.2, this satisfies indeed

$$(d \circ d)\alpha = 0 \quad \text{for any} \ \ \alpha \in \Omega^1_{q,N}.$$

It is easy to see that the map (2-80) is non–trivial. Moreover there is precisely one monomial of order 3 in the ξ variables, given by

$$\Theta^3 = -\frac{q^{-6}\Lambda_N r^2}{[2]_{q^2}[3]_q}\varepsilon^{ijk}\xi_i\xi_j\xi_k, \tag{2-81}$$

which commutes with all functions on the sphere,

$$[\Theta^3, f] = 0 \tag{2-82}$$

for all $f \in \mathcal{S}^2_{q,N}$. Finally, we complete the definition of the Hodge star operator by

$$*_H(1) = \Theta^3, \tag{2-83}$$

and by requiring that $(*_H)^2 = id$.

2.3.3 Star structure

A $*$–calculus (or a real form of $\Omega^*_{q,N}$) is a differential calculus which is a graded $*$–algebra such that the star preserves the grade, and satisfies [81]

$$\begin{aligned} (\alpha^{(n)}\alpha^{(m)})^* &= (-1)^{nm}(\alpha^{(m)})^*(\alpha^{(n)})^*, \\ (d\alpha^{(n)})^* &= d(\alpha^{(n)})^* \end{aligned} \tag{2-84}$$

for $\alpha^{(n)} \in \Omega^n_{q,N}$; moreover, the action of $U_q(su(2))$ must be compatible with the star on $U_q(su(2))$. Again, we have to distinguish the cases $q \in \mathbb{R}$ and $|q| = 1$.

1) $q \in \mathbb{R}$**.** In this case, the star structure must satisfy

$$(dx_i)^* = g^{ij}dx_j, \quad x_i^* = g^{ij}x_j, \tag{2-85}$$

which by (2-27) implies

$$\Theta^* = -\Theta. \tag{2-86}$$

Using (2-70), it follows that

$$\begin{aligned} \xi_i^* &= -g^{ij}\xi_j + q^2(q-q^{-1})\frac{[2]_q C_N}{r^2}g^{ij}dx_j \\ &= -g^{ij}\xi_j - q^2(q-q^{-1})\frac{[2]_q C_N}{r^2}g^{ij}\left(\Lambda_N\varepsilon_j^{kl}x_k\xi_l - (q-q^{-1})(qx_j\Theta - q^{-1}r^2\xi_j)\right). \end{aligned} \tag{2-87}$$

To show that this is indeed compatible with (2-60), one needs the following identity

$$q^2(q-q^{-1})\frac{[2]_q C_N}{r^2}(dx_i x_j - \hat{R}^{kl}_{ij}x_k dx_l) = (\mathbf{1} - (\hat{R}^2)^{kl}_{ij})\xi_k x_l \tag{2-88}$$

which can be verified with some effort, see Section 2.6.2. In particular, this shows that if one imposed $x_i\xi_j = (\hat{R}^{-1})^{kl}_{ij}\xi_k x_l$ instead of (2-60), one would obtain an equivalent calculus. This is unlike in the flat case, where one has two inequivalent calculi [85, 86]. Moreover, one can show that this real form is consistent with (2-71).

2) $|q| = 1$**.** In view of (2-42), it is easy to see that the star structure in this case is

$$(\xi_i)^* = q^{-4}\omega\xi_i\omega^{-1}, \quad x_i^* = -\omega x_i \omega^{-1}. \tag{2-89}$$

Recall that ω is a particular unitary element of $\mathcal{S}^2_{q,N}$ introduced in Section 2.2.3.

It is obvious using $(\hat{R}^{kl}_{ij})^* = (\hat{R}^{-1})^{lk}_{ji}$ that this is an involution which is consistent with (2-60), and one can verify that

$$\Theta^* = -\Theta. \tag{2-90}$$

This also implies

$$[\omega, \Theta] = 0,$$

hence

$$(dx_i)^* = -\omega dx_i \omega^{-1}. \tag{2-91}$$

Finally, $*_H$ is also compatible with the star structure:

$$(*_H(\alpha))^* = *_H(\alpha^*) \tag{2-92}$$

where $\alpha \in \Omega^1_{q,N}$, for both $q \in \mathbb{R}$ and $|q| = 1$. This is easy to see for $\alpha = \xi_i$ in the latter case, and for $\alpha = dx_i$ in the case $q \in \mathbb{R}$. This implies that indeed $(d\alpha^{(n)})^* = d(\alpha^{(n)})^*$ for all n.

We summarize the above results:

Theorem 2.3.1. *The definitions (2-78), (2-80) define a covariant differential calculus on* $\Omega^*_{q,N} = \oplus_{n=0}^3 \ \Omega^n_{q,N}$ *over* $\mathcal{S}^2_{q,N}$ *with* $dim(\Omega^n_{q,N}) = (1,3,3,1)$ *for* $n = (0,1,2,3)$. *Moreover, this is a* $*$*–calculus with the star structures (2-85) and (2-89) for* $q \in \mathbb{R}$ *and* $|q| = 1$*, respectively.*

2.3.4 Frame formalism

On many noncommutative spaces [75, 70], it is possible to find a particularly convenient set of one–forms (a “frame”) $\theta_a \in \Omega^1$, which commute with all elements in the function space Ω^0. Such a frame exists here as well, and in terms of the ξ_i variables, it takes a similar form to that of [75]. Consider the elements

$$\begin{aligned} \theta^a &= \Lambda_N \ S(L^{+a}_{\ \ j}) \ g^{jk}\xi_k \quad \in \Omega^1_{q,N}, & (2\text{-}93)\\ \lambda_a &= \frac{1}{\Lambda_N} \ x_i \ L^{+i}_{\ \ a} \qquad\quad \in \mathcal{S}^2_{q,N}. & (2\text{-}94)\end{aligned}$$

where as usual

$$\begin{aligned} L^{+i}_{\ \ j} &= \mathcal{R}_1 \pi^i_j(\mathcal{R}_2), & (2\text{-}95)\\ S(L^{+i}_{\ \ j}) &= \mathcal{R}_1^{-1}\pi^i_j(\mathcal{R}_2^{-1}) & (2\text{-}96)\end{aligned}$$

are elements of $U_q(su(2))$, which we consider here as elements in $\mathcal{S}^2_{q,N}$ via (2-33). Then the following holds:

Lemma 2.3.2.

$$
\begin{aligned}
[\theta^a, f] &= 0, && \text{(2-97)}\\
df &= [\lambda_a, f]\theta^a, && \text{(2-98)}\\
\Theta = x_i\xi_j g^{ij} &= \lambda_a\theta^a. && \text{(2-99)}
\end{aligned}
$$

for any $f \in \mathcal{S}^2_{q,N}$. In this sense, the λ_a are dual to the frame θ^b. They satisfy the relations

$$
\begin{aligned}
\lambda_a\lambda_b g^{ba} &= \frac{1}{q^4\Lambda_N^2}\, r^2,\\
\lambda_a\lambda_b\varepsilon^{ba}_c &= -\frac{1}{q^2}\,\lambda_c,\\
\theta^a\theta^b &= -q^2\,\hat{R}^{ba}_{cd}\theta^d\theta^c && \text{(2-100)}\\
d\theta^a &= \lambda_b[\theta^a,\theta^b]_+ + \frac{1}{q^2[2]_{q^2}}\varepsilon^a_{bc}\theta^c\theta^b\\
*_H\theta^a &= -\frac{1}{q^2[2]_{q^2}}\varepsilon^a_{bc}\theta^c\theta^b\\
\theta^a\theta^b\theta^c &= -\Lambda_N^2\frac{q^6}{r^2}\,\varepsilon^{cba}\Theta^3 && \text{(2-101)}
\end{aligned}
$$

In particular in the limit $q = 1$, this becomes $\lambda_a = \frac{1}{\Lambda_N}\, x_a$, and $dx_a = -\varepsilon^c_{ab}x_c\theta^b$, using (2-66).

Proof. Using

$$S(L^{+i}_{\ j})x_k = x_l(\hat{R}^{-1})^{ln}_{jk}S(L^{+i}_{\ n})$$

(which follows from (2-31)) and $\Delta(S(L^{+i}_{\ j})) = S(L^{+n}_{\ j}) \otimes S(L^{+i}_{\ n})$, it is easy to check that $[\theta^a, x_i] = 0$ for all i, a, and (2-97) follows. (2-99) follows immediately from $L^{+i}_{\ a}S(L^{+a}_{\ j}) = \delta^i_j$, and To see (2-101), one needs the well–known relation $L^{+l}_{\ r}L^{+k}_{\ s}g^{sr} = g^{kl}$, as well as $L^{+l}_{\ r}L^{+k}_{\ s}\varepsilon^{sr}_n = \varepsilon^{kl}_m L^{+m}_{\ n}$; the latter follows from the quasitriangularity of $U_q(su(2))$. The commutation relations among the θ are obtained as in [75] by observing

$$
\begin{aligned}
\theta^a\theta^b &= \Lambda_N\theta^a S(L^{+b}_{\ n})g^{nl}\xi_l\\
&= \Lambda_N S(L^{+b}_{\ n})\theta^a g^{nl}\xi_l\\
&= \Lambda_N^2 S(L^{+b}_{\ n})S(L^{+a}_{\ j})g^{jk}g^{nl}\xi_k\xi_l, && \text{(2-102)}
\end{aligned}
$$

using the commutation relations $\hat{R}^{kl}_{ij}SL^{+i}_{\ n}SL^{+j}_{\ m} = SL^{+k}_{\ i}SL^{+l}_{\ j}\hat{R}^{ij}_{nm}$, as well as (2-63). The remaining relations can be checked similarly. □

2.3.5 Integration of forms

As classically, it is natural to define the integral over the forms of the highest degree, which is 3 here. Since any $\alpha^{(3)} \in \Omega^3_{q,N}$ can be written in the from $\alpha^{(3)} = f\Theta^3$, we define

$$\int \alpha^{(3)} = \int f\Theta^3 := \int_{\mathcal{S}^2_{q,N}} f \qquad \text{(2-103)}$$

by (2-44), so that Θ^3 is the volume form. This definition is natural, since $[\Theta^3, f] = 0$. Integrals of forms with degree $\neq 3$ will be set to zero.

This integral satisfies an important cyclic property, as did the quantum trace (2-45). To formulate it, we extend the map S^2 from $\mathcal{S}^2_{q,N}$ to $\Omega^*_{q,N}$ by

$$S^2(\xi_i) = q^H \triangleright \xi_i,$$

extended as an algebra map. Then the following holds (see Section 2.6.2):

$$\int \alpha\ \beta = \int S^{-2}(\beta)\ \alpha \tag{2-104}$$

for any $\alpha, \beta \in \Omega^*_{q,N}$ with $deg(\alpha) + \deg(\beta) = 3$. Now Stokes theorem follows immediately:

$$\int d\alpha^{(2)} = \int [\Theta, \alpha^{(2)}] = 0 \tag{2-105}$$

for any $\alpha^{(2)} \in \Omega^2_{q,N}$, because $S^2\Theta = \Theta$. This purely algebraic derivation is also valid on some other spaces [87].

Finally we establish the compatibility of the integral with the star structure. From $\Theta^* = -\Theta$ and (2-47), we obtain

$$(\int \alpha^{(3)})^* = -\int (\alpha^{(3)})^* \qquad \text{for } \ q \in \mathbb{R}. \tag{2-106}$$

For $|q| = 1$, we have to extend the algebra map $\overline{f}$ (2-50) to $\Omega^*_{q,N}$. It turns out that the correct definition is

$$\overline{\xi_i} = -q^{-4}\ g^{ij}\xi_j + q^{-2}(q - q^{-1})\frac{[2]_q C_N}{r^2} g^{ij} dx_j, \tag{2-107}$$

extended as an antilinear algebra map; compare (2-87) for $q \in \mathbb{R}$. To verify that this is compatible with (2-60) and (2-71) requires the same calculations as to verify the star structure (2-87) for $q \in \mathbb{R}$. Moreover one can check using (2-69) that

$$\overline{dx_i} = -g^{ij} dx_j, \tag{2-108}$$

which implies that $\overline{\Theta} = \Theta$, and

$$\begin{aligned} \overline{*_H(\alpha)} &= *_H(\overline{\alpha}), \\ \overline{d\alpha} &= d\overline{\alpha}, \\ \overline{\overline{\alpha}} &= S^{-2}\alpha \end{aligned} \tag{2-109}$$

for any $\alpha \in \Omega^*_{q,N}$. Hence we have

$$(\int \alpha^{(3)})^* = \int \overline{\alpha^{(3)}} \qquad \text{for } \ |q| = 1. \tag{2-110}$$

2.4 Actions and fields

2.4.1 Scalar fields

With the tools provided in the previous sections, it is possible to construct actions for 2–dimensional euclidean field theories on the q–deformed fuzzy sphere.

We start with scalar fields, which are simply elements $\psi \in \mathcal{S}^2_{q,N}$. The obvious choice for the kinetic term is

$$\begin{aligned} S_{kin}[\psi] &= i\frac{r^2}{\Lambda_N^2}\int (d\psi)^* *_H d\psi \qquad \text{for } q \in \mathbb{R}, \\ S_{kin}[\psi] &= \frac{r^2}{\Lambda_N^2}\int \overline{d\psi} *_H d\psi \qquad \text{for } |q| = 1, \end{aligned} \tag{2-111}$$

which, using Stokes theorem, can equivalently be written in the form

$$\begin{aligned} S_{kin}[\psi] &= -i\frac{r^2}{\Lambda_N^2}\int \psi^*(d *_H d)\psi = -\frac{r^2}{\Lambda_N^2} i \int\limits_{\mathcal{S}^2_{q,N}} \psi^*(*_H d *_H d)\psi \qquad \text{for } q \in \mathbb{R}, \\ S_{kin}[\psi] &= -\frac{r^2}{\Lambda_N^2}\int \overline{\psi}(d *_H d)\psi = -\frac{r^2}{\Lambda_N^2} \int\limits_{\mathcal{S}^2_{q,N}} \overline{\psi}(*_H d *_H d)\psi \qquad \text{for } |q| = 1. \end{aligned} \tag{2-112}$$

They are real

$$S_{kin}[\psi]^* = S_{kin}[\psi] \tag{2-113}$$

for both $q \in \mathbb{R}$ and $|q| = 1$, using the reality properties established in the previous sections.

The fields can be expanded in terms of the irreducible representations

$$\psi(x) = \sum_{K,n} a^{K,n}\, \psi_{K,n}(x) \tag{2-114}$$

according to (2-30), with coefficients $a^{K,n} \in \mathbb{C}$; this corresponds to the first–quantized case. However, in order to ensure invariance of the actions under $U_q(su(2))$ (or a suitable subset thereof), we must assume that $U_q(su(2))$ acts on products of fields via the q–deformed coproduct. This can be implemented consistently only after a "second quantization", such that the coefficients in (2-114) generate a $U_q(su(2))$–module algebra. This will be presented in Chapter 5.

One can also consider real fields, which have the form

$$\begin{aligned} \psi(x)^* &= \psi(x) \qquad \text{for } q \in \mathbb{R}, \\ \overline{\psi(x)} &= \psi(x) \qquad \text{for } |q| = 1. \end{aligned} \tag{2-115}$$

This is preserved under the action of a certain real sector $\mathcal{G} \subset U_q(su(2))$ (2-140); the discussion is completely parallel to the one below (2-141) in the next section, hence we will not give it here.

Clearly $*_H d *_H d$ is the analog of the Laplace operator for functions, which can also be written in the usual form $d\delta + \delta d$, with $\delta = *_H d *_H$. It is hermitian by construction. We wish to evaluate it on the irreducible representations $\psi_K \in (2K+1)$, that is, on spin–K representations. The result is the following:

Lemma 2.4.1. *If $\psi_K \in \mathcal{S}^2_{q,N}$ is a spin K representation, then*

$$*_H d *_H d\psi_K = \frac{2}{[2]_q C_N} [K]_q [K+1]_q \ \psi_K. \tag{2-116}$$

The proof is in Section 2.6.2.

It is useful to write down explicitly the hermitian forms associated to the above kinetic action. Consider

$$\begin{aligned} S_{kin}[\psi, \psi'] &= i \frac{r^2}{\Lambda_N^2} \int (d\psi)^* *_H d\psi' \qquad \text{for } q \in \mathbb{R}, \\ S_{kin}[\psi, \psi'] &= \frac{r^2}{\Lambda_N^2} \int \overline{d\psi} *_H d\psi' \qquad \text{for } |q| = 1. \end{aligned} \tag{2-117}$$

Using Lemma 2.2.2, it follows immediately that they satisfy

$$\begin{aligned} S_{kin}[\psi, \psi']^* &= S_{kin}[\psi', \psi], \\ S_{kin}[\psi, u \triangleright \psi'] &= S_{kin}[u^* \triangleright \psi, \psi'] \end{aligned} \tag{2-118}$$

for both $q \in \mathbb{R}$ and $|q| = 1$. To be explicit, let $\psi_{K,n}$ be an orthonormal basis of (2K+1). We can be assume that it is a weight basis, so that n labels the weights from $-K$ to K. Then it follows that

$$S_{kin}[\psi_{K,n}, \psi_{K',m}] = c_K \ \delta_{K,K'} \ \delta_{n,m} \tag{2-119}$$

for some $c_K \in \mathbb{R}$. Clearly one can also consider interaction terms, which could be of the form

$$S_{int}[\psi] = \int_{\mathcal{S}^2_{q,N}} \psi\psi\psi, \tag{2-120}$$

or similarly with higher degree.

2.4.2 Gauge fields

Gauge theories arise in a very natural way on $\mathcal{S}^2_{q,N}$. For simplicity, we consider only the analog of the abelian gauge fields here. They are simply one–forms

$$B = \sum B_a \theta^a r \quad \in \Omega^1_{q,N}, \tag{2-121}$$

which we expand in terms of the frames θ^a introduced in Section 2.3.4. Notice that they have 3 independent components, which reflects the fact that calculus is 3–dimensional. Loosely speaking, the fuzzy sphere does see a shadow of the 3–dimensional embedding space. One of the components is essentially radial and should be considered as a scalar field, however it is naturally tied up with the other 2 components of B. We will impose the reality condition

$$\begin{aligned} B^* &= B \qquad \text{for } q \in \mathbb{R}, \\ \overline{B} &= q^{-\frac{H}{2}} \triangleright B \qquad \text{for } |q| = 1. \end{aligned} \tag{2-122}$$

Since only 3–forms can be integrated, the most simple candidates for Langrangians that can be written down have the form

$$S_3 = \frac{1}{r^2\Lambda_N^2}\int B^3, \quad S_2 = \frac{1}{r^2\Lambda_N^2}\int B *_H B, \quad S_4 = \frac{1}{r^2\Lambda_N^2}\int B^2 *_H B^2. \tag{2-123}$$

They are clearly real, with the reality condition (2-122); the factor i for real q is omitted here. We also define

$$F := B^2 - *_H B, \tag{2-124}$$

for reasons which will become clear below. The meaning of the field B becomes obvious if one writes it in the form

$$B = \Theta + A, \qquad B_a = \frac{1}{r}\lambda_a + A_a \tag{2-125}$$

While B and Θ become singular in the limit $N \to \infty$, A remains well–defined. Using

$$\begin{aligned} F &= dA + A^2, \\ \int A\Theta^2 &= \int dA\Theta = \int *_H A\, \Theta, \\ \int A^2\Theta &= \frac{1}{2}\int (AdA + A *_H A) \end{aligned} \tag{2-126}$$

which follow from (2-78), one finds

$$\begin{aligned} S_2 &= \frac{1}{r^2\Lambda_N^2}\int A *_H A + 2A\Theta^2 \\ S_3 &= \frac{1}{r^2\Lambda_N^2}\int A^3 + \frac{3}{2}(AdA + A *_H A) + 3A\Theta^2 + \Theta^3 \end{aligned} \tag{2-127}$$

and

$$\boxed{S_{YM} := \frac{1}{r^2\Lambda_N^2}\int F *_H F = \frac{1}{r^2\Lambda_N^2}\int (dA + A^2) *_H (dA + A^2).} \tag{2-128}$$

The latter action (which is a linear combination of S_2, S_3, and S_4) is clearly the analog of the Yang–Mills action, which in the classical limit contains a gauge field an a scalar, as we will see below. In the limit $q \to 1$, it reduces to the action considered in [88].

The actions S_3 and S_2 alone contain terms which are linear in A, which would indicate that the definition of A (2-125) is not appropriate. However, the linear terms cancel in the following linear combination

$$\boxed{S_{CS} := \tfrac{1}{3}S_3 - \tfrac{1}{2}S_2 = -\tfrac{2\pi}{3\Lambda_N^2} + \tfrac{1}{2}\tfrac{1}{r^2\Lambda_N^2}\int AdA + \tfrac{2}{3}A^3.} \tag{2-129}$$

Notice that the "mass term" $A *_H A$ has also disappeared. This form is clearly the analog of the Chern–Simons action. It is very remarkable that it exists on $\mathcal{S}^2_{q,N}$, which is related to the fact that the calculus is 3–dimensional. In the case $q=1$, this is precisely what has been found recently in the context of 2–branes on the $SU(2)$ WZW model [30].

In terms of the components (2-121), $B^2 = B_a B_b \theta^a\theta^b r^2$, and $*_H B = -\frac{r}{q^2[2]_{q^2}} B_a \varepsilon^a_{bc}\theta^c\theta^b$. Moreover, it is easy to check that

$$\begin{aligned} *_H(\theta^b\theta^c) &= -q^2\, \varepsilon^{cb}_a \theta^a, \\ \theta^a *_H \theta^b &= \Lambda_N^2 \frac{q^4}{r^2} g^{ba}\, \Theta^3, \\ \theta^a\theta^b *_H \theta^c\theta^d &= [2]_{q^2}\Lambda_N^2 \frac{q^8}{r^2}(P^-)^{dc}_{a'b'}\, g^{b'b}g^{a'a}\, \Theta^3 = \Lambda_N^2\frac{q^8}{r^2}\varepsilon^{dc}_n\varepsilon^{ba}_m g^{nm}\Theta^3. \end{aligned} \tag{2-130}$$

Hence

$$\begin{aligned} F &= (B_aB_b + \frac{1}{q^2r[2]_{q^2}}B_c\, \varepsilon^c_{ba})\theta^a\theta^b r^2 = (\frac{\lambda_a}{r}A_b + A_a\frac{\lambda_b}{r} + A_aA_b + \frac{1}{q^2r[2]_{q^2}}A_c\, \varepsilon^c_{ba})\theta^a\theta^b r^2 \\ &= F_{ab}\, \theta^a\theta^b r^2, \end{aligned} \tag{2-131}$$

where we define F_{ab} to be totally antisymmetric, i.e. $F_{ab} = (P^-)^{b'a'}_{ba}F_{a'b'}$ using (2-100). This yields

$$S_{YM} = q^8[2]_{q^2}\int\limits_{\mathcal{S}^2_{q,N}} F_{ab}F_{cd}\, (P^-)^{dc}_{a'b'}\, g^{b'b}g^{a'a}, \tag{2-132}$$

To understand these actions better, we write the gauge fields in terms of "radial" and "tangential" components,

$$A_a = \frac{x_a}{r}\phi + A^t_a \tag{2-133}$$

where ϕ is defined such that

$$x_aA^t_b\, g^{ab} = 0; \tag{2-134}$$

this is always possible. However to get a better insight, we consider the case $q=1$, and take the classical limit $N\to\infty$ in the following sense: for a given (smooth) field configuration in $\mathcal{S}^2_N$, we use the sequence of embeddings of $\mathcal{S}^2_{q,N}$ to approximate it for $N\to\infty$. Then terms of the form $[A^t_a, A^t_b]$ vanish in the limit (since the fields are smooth in the limit). The curvature then

splits into a tangential and radial part, $F_{ab} = F^t_{ab} + F^\phi_{ab}$, where[4]

$$\begin{aligned} F^t_{ab} &= \frac{1}{2r}\left([\lambda_a, A^t_b] - [\lambda_b, A^t_a] + A^t_c\, \varepsilon^c_{ba}\right), \\ F^\phi_{ab} &= \frac{1}{2r^2}\left(\varepsilon^c_{ab} x_c \phi + [\lambda_a, \phi] x_b - [\lambda_b, \phi] x_a\right). \end{aligned} \tag{2-135}$$

Moreover,

$$\begin{aligned} x^a F^t_{ab} &\to \frac{1}{4r}[x^a \lambda_a, A^t_b] - \frac{1}{2r}[\lambda_b, x^a A^t_a] = 0, \\ x^a[\lambda_a, \phi] &\to \frac{1}{2}[x^a \lambda_a, \phi] = 0 \end{aligned} \tag{2-136}$$

in the classical limit, which implies that

$$\begin{aligned} \int\limits_{S^2} F^t_{ab} F^{\phi ab} &= \int\limits_{S^2} \frac{1}{2r^2} \varepsilon^n_{ab} x_n \phi F^{t\,ab}, \\ \int\limits_{S^2} F^\phi_{ab} F^{\phi ab} &= \int\limits_{S^2} \frac{1}{2r^2}\left(\phi^2 + [\lambda_a, \phi][\lambda^a, \phi]\right) \end{aligned}$$

in the limit. Therefore we find

$$\boxed{S_{YM} = -\int\limits_{S^2} \left(2F^t_{ab} F^t{}_{ab} + \tfrac{2}{r^2} \varepsilon^n_{ab} x_n \phi F^{t\,ab} + \tfrac{1}{r^2}(\phi^2 + [\lambda_a, \phi][\lambda^a, \phi])\right)} \tag{2-137}$$

in the limit, as in [88]. Notice that in the flat limit $r \to \infty$, the $F - \phi$ coupling term vanishes. Similarly, the Chern–Simons action (2-129) becomes

$$\begin{aligned} S_{CS} &\to -\frac{2\pi}{3\Lambda_N^2} + \frac{1}{2r^2 \Lambda_N^2} \int dA^t (A^t + 2\Lambda_N \Theta \phi) - \Lambda_N^2 \phi^2 \Theta^3 \\ &= -\frac{2\pi}{3\Lambda_N^2} + \frac{1}{2r} \int\limits_{S^2} F^t_{ab}(A^t_c + 2\frac{x_c}{r}\phi)\varepsilon^{abc} - \frac{1}{2r^2} \int\limits_{S^2} \phi^2 \end{aligned} \tag{2-138}$$

for $N \to \infty$. In the flat limit $r \to \infty$, the term $F^t_{ab} A^t_c \varepsilon^{abc}$ vanishes because of (2-136), leaving the $F - \phi$ coupling term (after multiplying with r).

Back to finite N and $q \neq 1$. To further justify the above definition of curvature (2-124), we consider the zero curvature condition, $F = 0$. In terms of the B fields, this is equivalent to

$$\varepsilon^{ba}_c B_a B_b + \frac{1}{q^2 r} B_c = 0 \tag{2-139}$$

which is (up to rescaling) the same as equation (2-26) with *opposite* multiplication[5]; in particular, the solutions $B_a \in \mathcal{S}^2_{q,N}$ are precisely all possible representations of $U_q^{op}(su(2))$ in the space

[4] the pull–back of F to the 2–sphere in the classical case is unaffected by this split

[5] this can be implemented e.g. using the antipode of $U_q(su(2))$

of $N+1$–dimensional matrices. They are of course classified by the number of partitions of $Mat(N+1)$ into blocks with sizes $n_1, ..., n_k$ such that $\sum n_i = N+1$, as in the case $q = 1$.

These solutions can be interpreted as fuzzy spheres of various sizes. Quite remarkably, this shows that gauge theory on the fuzzy sphere (q-deformed or not) is more that we anticipated: it describes in fact a gauge theory on various (superpositions of) fuzzy spheres simultaneously, and one can expect that even transitions between different spheres can occur. This is indeed the case, and will be described in Chapter 3 in the undeformed case. This is very much in accord with the picture of D-branes on group manifolds as described in [30].

Gauge invariance. We have seen that actions which describe gauge theories in the limit $q = 1$ arise very naturally on $\mathcal{S}^2_{q,N}$ (as on certain other higher–dimensional q–deformed spaces [89]). However, it is less obvious in which sense they are actually gauge–invariant for $q \neq 1$. For $q = 1$, the appropriate gauge transformation is $B \to UBU^{-1}$, for a unitary element $U \in \mathcal{S}^2_N$. This transformation does not work for $q \neq 1$, because of (2-104). Instead, we propose the following: let

$$\begin{aligned}\mathcal{H} &= \{\gamma \in U_q(su(2)): \ \ \varepsilon(\gamma) = 0, \ \gamma^* = S\gamma\}, \\ \mathcal{G} &= \{\gamma \in U_q(su(2)): \ \ \varepsilon(\gamma) = 1, \ \gamma^* = S\gamma\} = e^{\mathcal{H}},\end{aligned} \tag{2-140}$$

for $q \in \mathbb{R}$; for $|q| = 1$, the $S\gamma$ on the rhs should be replaced by $S_0(\gamma) = q^{-\frac{H}{2}} S(\gamma) q^{\frac{H}{2}}$. Clearly $\mathcal{H}$ is a subalgebra (without 1) of $U_q(su(2))$, and $\mathcal{G}$ is closed under multiplication. Using the algebra map j (2-33), $\mathcal{H}$ can be mapped to some real sector of the space of functions on the fuzzy sphere.

Now consider the following "gauge" transformations:

$$B \to j(\gamma_{(1)}) B j(S\gamma_{(2)}) \qquad \text{for } \ \gamma \in \mathcal{G}. \tag{2-141}$$

It can be checked easily that these transformations preserve the reality conditions (2-122) for both real q and $|q| = 1$. In terms of components $B = B_a \theta^a r$, this transformation is simply (suppressing j)

$$B_a \to \gamma_{(1)} B_a S\gamma_{(2)} = \gamma \triangleright B_a, \tag{2-142}$$

which is the rotation of the fields $B_a \in \mathcal{S}^2_{q,N}$ considered as scalar fields[6], i.e. the rotation $\gamma \in U_q(su(2))$ does not affect the index a because of (2-97). In terms of the A_a variables, this becomes

$$A_a \theta^a \to \gamma_{(1)} A_a S\gamma_{(2)} \theta^a + \gamma_{(1)} d(S\gamma_{(2)}) = (\gamma \triangleright A_a)\theta^a + \gamma_{(1)} d(S\gamma_{(2)}), \tag{2-143}$$

using (2-65) and (2-97). Hence these transformations are a mixture of rotations of the components (first term) and "pure gauge transformations" (second term). Moreover, the radial and tangential components get mixed.

To understand these transformations better, consider $q = 1$. Then we have two transformations of a given gauge field B_a, the first by conjugation with an unitary element $U \in \mathcal{S}^2_{q,N}$, and

[6]notice that this is *not* the rotation of the one–form B, because $\gamma_{(1)} \xi_i S\gamma_{(2)} \neq \gamma \triangleright \xi_i$

the second by (2-141). We claim that the respective spaces of inequivalent gauge fields are in fact equivalent. Indeed, choose e.g. $a = 1$; then there exists a unitary $U \in \mathcal{S}^2_{q,N}$ such that $U^{-1} B_a U$ is a diagonal matrix with real entries. On the other hand, using a suitable $\gamma \in \Gamma$ and recalling (2-30), one can transform B_a into the form $B_a = \sum_i b_i (x_0)^i$ with real b_i, which is again represented by a diagonal matrix in a suitable basis. Hence at least generically, the spaces of inequivalent gauge fields are equivalent.

One can also see more intuitively that (2-143) corresponds to an abelian gauge transformation in the classical limit. Consider again $\gamma(x) = e^{ih(x)}$ with $h(x)^* = -Sh(x)$, approximating a smooth function in the limit $N \to \infty$. Using properly rescaled variables x_i, one can see using (2-32) that if viewed as an element in $U(su(2))$, γ approaches the identity, that is $\gamma \triangleright A_a(x) \to A_a(x)$ in the classical limit. Now write the functions on $\mathcal{S}^2_N$ in terms of the variables x_1 and x_{-1}, for example. Then (2-32) yields

$$(\mathbf{1} \otimes S)\Delta(x_i) = x_i \otimes 1 - 1 \otimes x_i, \tag{2-144}$$

for $i = \pm 1$, and one can see that

$$\gamma_{(1)}[\lambda_i, S\gamma_{(2)}] \approx \partial_i h(x_i) \tag{2-145}$$

in the (flat) classical limit. Hence (2-143) indeed becomes a gauge transformation in the classical limit.

To summarize, we found that the set of gauge transformations in the noncommutative case is a (real sector of a) quotient of $U_q(su(2))$, and can be identified with the space of (real) functions on $\mathcal{S}^2_{q,N}$ using the map j. However, the transformation of products of fields is different from the classical case. Classically, the gauge group acts on products "componentwise", which means that the coproduct is trivial. Here, we must assume that $U_q(su(2))$ acts on products of fields via the q–deformed coproduct, so that the above actions are invariant under gauge transformations, by (2-46). In particular, the "gauge group" has become a real sector of a Hopf algebra. Of course, this can be properly implemented on the fields only after a "second quantization", as in the case of rotation invariance (see Section 2.4.1). This will be presented in Chapter 5. This picture is also quite consistent with observations of a BRST–like structure in $U_q(so(2,3))$ at roots of unity, see [89, 90].

Finally, we point out that the above actions are invariant under a global $U_q(su(2))$ symmetry, by rotating the frame θ^a.

2.5 Discussion

In this chapter, we studied the q–deformed fuzzy sphere $\mathcal{S}^2_{q,N}$, which is an associative algebra which is covariant under $U_q(su(2))$, for real q and q a phase. In the first case, this is the same as the "discrete series" of Podleś quantum spheres. We developed the formalism of differential forms and frames, as well as integration. We then briefly considered scalar and gauge field theory

on this space. It appears that $\mathcal{S}^2_{q,N}$ is a nice and perhaps the simplest example of a quantum space which is covariant under a quantum group. This makes it particularly well suited for studying field theory, which has proved to be rather difficult on other q–deformed spaces. We were able to write hermitian actions for scalar and gauge fields, including analogs of Yang–Mills and Chern–Simons actions. In particular, the form of the actions for gauge theories suggests a new type of gauge symmetry, where the role of the gauge group is played by $U_q(su(2))$, which can be mapped onto the space of functions on $\mathcal{S}^2_{q,N}$. This suggests that formulating field theory on quantized spaces which are not based on a Moyal-Weyl star product is qualitatively different, and may lead to interesting new insights.

To put all this into perspective, we recall that the main motivation for considering $\mathcal{S}^2_{q,N}$ was the discovery [8] that a quasi–associative twist of $\mathcal{S}^2_{q,N}$ arises on spherical D–branes in the $SU(2)_k$ WZW model, for q a root of unity. In view of this, we hope that the present formalism may be useful to formulate a low–energy effective field theory induced by open strings ending on the D–branes. This in turn suggests to consider also the second quantization of field theories on $\mathcal{S}^2_{q,N}$, corresponding to a loop expansion and many–particle states. It is quite interesting that even from a purely formal point of view, a second quantization seems necessary for a satisfactory implementation of the q-deformed symmetries in such a field theory. This is the subject of Chapter 5. Finally, while the question of using either the quasi–associative algebra (1-18) or the associative $\mathcal{S}^2_{q,N}$ (1-19) may ultimately be a matter of taste, the latter does suggest certain forms for Lagrangians, induced by the differential calculus. It would be very interesting to compare this with a low–energy effective action induced from string theory beyond the leading term in a $1/k$ expansion.

2.6 Technical complements to Chapter 2

2.6.1 Invariant tensors for $U_q(su(2))$

The general properties of invariant tensors were explained in Section 1.4.2. The q–deformed epsilon–symbol ("spinor metric") for spin 1/2 representations is given by

$$\varepsilon^{+-} = q^{\frac{1}{2}}, \;\; \varepsilon^{-+} = -q^{-\frac{1}{2}}, \tag{2-146}$$

all other components are zero. The corresponding tensor with lowered indices is $\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta}$ and satisfies $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. In particular, $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = -(q+q^{-1}) = -[2]_q$.

The q–deformed sigma–matrices, i.e. the Clebsch–Gordon coefficients for $(3) \subset (2) \otimes (2)$, are given by

$$\begin{aligned} \sigma_1^{++} &= 1 = \sigma_{-1}^{--}, \\ \sigma_0^{+-} &= \frac{q^{-\frac{1}{2}}}{\sqrt{[2]_q}}, \;\; \sigma_0^{-+} = \frac{q^{\frac{1}{2}}}{\sqrt{[2]_q}} \end{aligned} \tag{2-147}$$

in an orthonormal basis, and $\sigma_i^{\alpha\beta} = \sigma^i_{\alpha\beta}$. They are normalized such that $\sum_{\alpha\beta} \sigma^i_{\alpha\beta}\sigma_j^{\alpha\beta} = \delta^i_j$. That is, they define a unitary map (at least for $q \in \mathbb{R}$).

The q–deformed invariant tensor for spin 1 representations is given by

$$g^{1-1} = -q, \quad g^{00} = 1, \quad g^{-11} = -q^{-1}, \tag{2-148}$$

all other components are zero. Then $g_{\alpha\beta} = g^{\alpha\beta}$ satisfies $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$, and $g^{\alpha\beta}g_{\alpha\beta} = q^2 + 1 + q^{-2} = [3]_q$.

The Clebsch–Gordon coefficients for $(3) \subset (3)\otimes(3)$, i.e. the q–deformed structure constants, are given by

$$\begin{array}{ll} \varepsilon_1^{10} = q, & \varepsilon_1^{01} = -q^{-1}, \\ \varepsilon_0^{00} = q - q^{-1}, & \varepsilon_0^{1-1} = 1 = -\varepsilon_0^{-11}, \\ \varepsilon_{-1}^{0-1} = q, & \varepsilon_{-1}^{-10} = -q^{-1} \end{array} \tag{2-149}$$

in an orthonormal basis, and $\varepsilon^k_{ij} = \varepsilon_k^{ij}$. They are normalized such that $\sum_{ij} \varepsilon^n_{ij}\varepsilon^{ij}_m = [2]_{q^2}\delta^n_m$. Moreover, the following identities hold:

$$\varepsilon^n_{ij} g^{jk} = \varepsilon_i^{nk} \tag{2-150}$$

$$g_{ij}\varepsilon^j_{kl} = \varepsilon^j_{ik} g_{jl} \tag{2-151}$$

$$\varepsilon_i^{nk}\varepsilon_k^{lm} - \varepsilon_i^{km}\varepsilon_k^{nl} = g^{nl}\delta_i^m - \delta_i^n g^{lm} \tag{2-152}$$

which can be checked explicitly. In view of (2-151), the q–deformed totally (q–)antisymmetric tensor is defined as follws:

$$\varepsilon^{ijk} = g^{in}\varepsilon_n^{jk} = \varepsilon_n^{ij} g^{nk}. \tag{2-153}$$

It is invariant under the action of $U_q(su(2))$.

2.6.2 Some proofs

Proof of (2-66): Using the identity

$$\mathbf{1} = q^{-2}\hat{R} + (1+q^{-4})P^- + (1-q^{-6})P^0, \tag{2-154}$$

(2-150), (2-63), and the braiding relation (2-60) we can calculate the commutation relation of Θ with the generators x_i:

$$\begin{aligned} x_i\Theta &= x_i(x_j\xi_t g^{jt}) \\ &= q^{-2}\hat{R}^{kl}_{ij}x_k x_l \xi_t g^{jt} + q^{-2}\Lambda_N \varepsilon^n_{ij} x_n \xi_t g^{jt} + \frac{r^2}{[3]}(1-q^{-6}) g_{ij}\xi_t g^{jt} \\ &= q^{-2}\Theta x_i + q^{-2}\Lambda_N \varepsilon^n_{ij} x_n \xi_t g^{jt} + r^2\frac{(1-q^{-6})}{[3]_q}\xi_i \\ &= q^{-2}\Theta x_i + q^{-2}\Lambda_N \varepsilon_i^{nk} x_n \xi_k + r^2 q^{-3}(q-q^{-1})\xi_i, \end{aligned}$$

which yields (2-66).

Proof of (2-72) and (2-73): Using (2-71), one has

$$\Theta\xi_i = -q^2\xi_i\Theta, \tag{2-155}$$

which implies $\Theta\Theta = \Theta(x\cdot\xi) = dx\cdot\xi - q^2\Theta\Theta$, hence

$$(1+q^2)\Theta^2 = dx\cdot\xi.$$

On the other hand, (2-66) yields

$$dx\cdot\xi = -\Lambda_N x_i\varepsilon_j^{kl}\xi_k\xi_l g^{ij} - q^3(q-q^{-1})\Theta^2,$$

and combining this it follows that

$$\Theta^2 = -\frac{q^2\Lambda_N}{[2]_{q^2}}x_i\xi_k\xi_l\varepsilon^{ikl}.$$

We wish to relate this to $dx_i dx_j g^{ij}$, which is proportional to $d\Theta$. Using the relations $\varepsilon_i^{nk}x_n\xi_k = -q^{-2}\varepsilon_i^{nk}\xi_n x_k$, $\Theta = q^{-4}\xi\cdot x$, (2-151) and (2-152), one can show that

$$\varepsilon_i^{nk}x_n\xi_k\varepsilon_j^{ml}x_m\xi_l g^{ij} = \Lambda_N x_i\xi_k\xi_l\varepsilon^{ikl} + q^2\Theta^2$$

which using (2-66) implies

$$dx_i dx_j g^{ij} = -\frac{1}{C_N}\ r^2\ \Theta^2.$$

Proof of $d\circ d = 0$ **on** $\Omega^1_{q,N}$**.** First, we calculate

$$\begin{aligned}
[\Theta, d\xi_i] &= (q^2-1)(q^2+1)\left(\xi_i\Theta^2 + \frac{q^{-2}\Lambda_N}{[2]_{q^2}}\varepsilon_i^{kl}\xi_k\xi_l\Theta\right)\\
&= (q^2-1)(q^2+1)\left(\xi_i(*_H\Theta) - (*_H\xi_i)\Theta\right) = 0
\end{aligned}$$

using (2-155), (2-76), and (2-75). This implies that

$$\begin{aligned}
d(d(f\xi_i)) &= [\Theta, df\xi_i + fd\xi_i]\\
&= [\Theta, df]_+\ \xi_i - df[\Theta,\xi_i]_+ + df d\xi_i + f[\Theta, d\xi_i]\\
&= ddf + *_H(df)\xi_i - df(d\xi_i + *_H(\xi_i)) + df d\xi_i\\
&= -df *_H \xi_i + (*_H df)\xi_i = 0
\end{aligned}$$

by (2-75) for any $f\in\mathcal{S}^2_{q,N}$. This proves $d\circ d = 0$ on $\Omega^1_{q,N}$.

Proof of (2-75). First, we show that

$$(*_H\xi_i)\xi_j = \xi_i(*_H\xi_j), \tag{2-156}$$

which is equivalent to

$$\varepsilon_i^{nk}\xi_n\xi_k\xi_j = \xi_i\varepsilon_j^{nk}\xi_n\xi_k.$$

Now $\Omega^3_{q,N}$ is one–dimensional as module over $\mathcal{S}^2_{q,N}$, generated by Θ^3 (2-81), which in particular is a singlet under $U_q(su(2))$. This implies that

$$\begin{aligned}
\varepsilon_i^{nk}\xi_n\xi_k\xi_j &= (P^0)_{ij}^{rs}\varepsilon_r^{nk}\xi_n\xi_k\xi_s \\
&= -\frac{q^6[2]_{q^2}}{\Lambda_N r^2}\; g_{ij}\;\Theta^3 \\
&= (P^0)_{ij}^{rs}\xi_r\varepsilon_s^{nk}\xi_n\xi_k = \xi_i\varepsilon_j^{nk}\xi_n\xi_k,
\end{aligned} \tag{2-157}$$

as claimed. Now (2-75) follows immediately using the fact that $*_H$ is a left–and right $\mathcal{S}^2_{q,N}$–module map.

Reality structure for $q \in \mathbb{R}$: These are the most difficult calculations, and they are needed to verify (2-107) as well. First, we have to show that (2-60) is compatible with the star structure (2-87). By a straightforward calculation, one can reduce the problem to proving (2-88). We verify this by projecting this quadratic equation to its spin 0, spin 1, and spin 2 part. The first two are easy to check, using (2-69) in the spin 1 case. To show the spin 2 sector, it is enough to consider (2-88) for $i = j = 1$, by covariance. This can be seen e.g. using $[x_1, \varepsilon_1^{ij}x_i\xi_j] = -q^{-2}\Lambda_N x_1\xi_1$, which in turn can be checked using (2-154), (2-62) and (2-149).

Next, we show that (2-71) is compatible with the star structure (2-87). This can be reduced to

$$(q^2\hat{R} - q^{-2}\hat{R}^{-1})_{ij}^{kl}\; dx_k\;\xi_l = q^2(q-q^{-1})\frac{[2]_q C_N}{r^2}(\mathbf{1} + q^2\hat{R})_{ij}^{kl})\; dx_k\; dx_l$$

The spin 0 part is again easy to verify, and the spin 1 part vanishes identically (since then $\hat{R}$ has eigenvalue $-q^{-2}$). For the spin 2 part, one can again choose $i = j = 1$, and verify it e.g. by comparing with the differential of equation (2-88).

Proof of (2-104): Since $\Omega^*_{q,N}$ is finitely generated and because of (2-45) and $[\Theta^3, f] = 0$, it is enough to consider $\beta = \xi_k$. In this case, the claim reduces to

$$\xi_i\xi_j\xi_k = S^{-2}(\xi_k)\xi_i\xi_j.$$

Now $S^{-2}(\xi_k) = D_k^l\xi_l$, where $D_k^l = \delta_k^l q^{2r_l}$ with $r_l = (-2, 0, 2)$ for $l = (1, 0, -1)$, respectively. Since $\xi_i\xi_j = \frac{1}{[2]_{q^2}}\;\varepsilon_{ij}^n(\varepsilon_n^{rs}\xi_r\xi_s)$, there remains to show that $(\varepsilon_n^{rs}\xi_r\xi_s)\xi_k = S^{-2}(\xi_k)(\varepsilon_n^{rs}\xi_r\xi_s)$. By (2-157), this is equivalent to

$$g_{nk}\Theta^3 = D_k^l g_{ln}\Theta^3,$$

which follows from the definition of D_k^l.

Proof of Lemma 2.4.1: Using (2-77), (2-82) and $d\Theta^2 = 0$, we have

$$\begin{aligned} d *_H d\psi &= d(\psi\Theta^2 - \Theta^2\psi) = (d\psi)\Theta^2 - \Theta^2 d\psi \\ &= (d\psi)\Theta^2 + [\Theta^2, \psi]\Theta \\ &= (d\psi)\Theta^2 + (*_H d\psi)\Theta. \end{aligned} \tag{2-158}$$

To proceed, we need to evaluate $d\psi_K$. Because it is an irreducible representation, it is enough to consider $\psi_K = (x_1)^K$. From (2-88) and using $\xi_1 x_1 = q^{-2} x_1 \xi_1$, it follows that

$$dx_1 x_1 = q^2 x_1 dx_1 - \frac{q^{-2}}{C_N} r^2 x_1 \xi_1,$$

since $\hat{R}$ can be replaced by q^2 here. By induction, one finds

$$dx_1 x_1^k = x_1^k \left(q^{2k} dx_1 - [k]_{q^2} \frac{q^{-2}}{C_N}\, r^2 \xi_1 \right), \tag{2-159}$$

and by an elementary calculation it follows that

$$d(x_1^{k+1}) = [k+1]_q x_1^k \left(q^k dx_1 - \frac{q^{-2}}{[2]_q C_N} [k]_q r^2 \xi_1 \right). \tag{2-160}$$

Moreover, we note that using (2-75)

$$(\xi_i \Theta + *_H \xi_i)\Theta = \xi_i(*_H\Theta) + (*_H\xi_i)\Theta = 2(*_H\xi_i)\Theta = -\frac{2}{r^2} x_i \Theta^3. \tag{2-161}$$

The last equality follows easily from (2-157) and (2-155). Similarly

$$(dx_i\Theta + *_H dx_i)\Theta = 2 *_H dx_i \Theta = 2 dx_i \Theta^2. \tag{2-162}$$

Now we can continue (2-158) as

$$\begin{aligned} d *_H dx_1^K &= (dx_1^{K-1}\,\Theta + *_H dx_1^{K-1})\Theta \\ &= [K]_q x_1^{K-1} \left(2q^{K-1} dx_1 \Theta^2 - 2\frac{q^{-2}}{[2]_q C_N} [K-1]_q x_1 \Theta^3 \right). \end{aligned} \tag{2-163}$$

Finally it is easy to check that

$$dx_i \Theta^2 = -\frac{1}{C_N} x_i \Theta^3, \tag{2-164}$$

and after a short calculation one finds (2-116).

Chapter 3

Fuzzy instantons

To some extent, the ideas behind the construction of instantons and solitons can be applied to the noncommutative geometries under consideration here. For our purpose, an instanton is a finite–action solution of Yang–Mills type equations of motion with euclidean signature, such that the field equations reduce to self–duality conditions. The word soliton refers to a stable finite-energy solution. In ordinary geometry they are both stable because of a topological obstruction to their decay. The notion of topology now becomes somewhat “fuzzy”; nevertheless such solutions exist in our context, and it is interesting to study their structure on a fuzzy space. This chapter is based on a joint work [37] with Harald Grosse, Marco Maceda and John Madore, however with some changes especially in the scaling behaviour of the sphere, adapting it to the D-brane scenario of the preceding sections.

We add in this chapter a commuting (euclidean) time coordinate to the undeformed fuzzy sphere S^2_N, and consider a Maxwell–like $U(1)$ gauge theory [29, 88]. The spacetime is hence 3-dimensional, and it may seem impossible to have an instanton, which is a self-dual solution of some equation of motion. Nevertheless, this is possible here. The reason is that the fuzzy sphere S^2_N sees some trace of the 3-dimensional embedding space. We already noticed this in Chapter 2, where the calculus on the fuzzy sphere (whether or not it is q-deformed) turned out to be 3-dimensional. Hence with an extra time it is 4-dimensional, and admits a notion of self-duality.

From a physical point of view, the most important use of instantons is to describe tunneling between different topological sectors of a gauge theory. In our situation, they turn out to play a similar role: they interpolate between different geometrical vacua of the theory. We already encountered these vacua in Chapter 2, as block-type solutions of the zero curvature condition in Section 2.4.2. They describe in the classical limit spheres of different sizes, or superpositions thereof. They can be viewed as ‘fuzzy’ spherical $D2$-branes, and were already found in [91]. The instantons now tunnel between two such $D2$-brane configurations, for example one in which the branes coincide and the other in which they are completely separated. Clearly, these instantons disappear in the classical limit; however the quantum nature of fuzzy geometry admits such tunneling between different classical geometries.

We stress here the role of instantons as mediators between different stable vacuum sectors of

a matrix geometry. Several other aspects of instantons have also been carried over [92, 93, 94, 95, 96] into various noncommutative geometries including the fuzzy sphere [97, 98, 99]. Similar calculations have been carried out [100] on the torus.

3.1 Abelian gauge theory on $S^2_N \times \mathbb{R}$

3.1.1 The noncommutative geometry.

We recall very briefly some formulae from Section 2.3 which will be useful below. The algebra $\mathcal{A}$ of functions on $S^2_N \times \mathbb{R}$ is generated by generators λ_a which satisfy the commutation relations

$$[\lambda_a, \lambda_b] = \varepsilon^c{}_{ab}\lambda_c, \quad \lambda_a\lambda^a = \frac{N(N+2)}{4}, \tag{3-1}$$

together with a commutative time coordinate t. Let θ^a be a real frame [68, 69, 101] for the differential calculus over the fuzzy sphere, and let $\theta^0 = dt$ be the standard de Rham differential along the real line. The differential df of $f \in \mathcal{A}$ can then be written (by definition) as

$$df = [\lambda_a, f]\theta^a + \dot{f}\theta^0, \qquad \dot{f} = \partial_t f,$$

The λ_a are considered dimensionless here. We use the basis of anti-commuting one-forms $\theta^\alpha = (\theta^0, \theta^a)$ for $a = 1, 2, 3$. The form $\theta = \lambda_a\theta^a$ generates the "spatial part" of the exterior derivative, and can be considered as a flat connection (2-73). The exterior derivative of a 2-form is given by the obvious adaptation of (2-78),

$$\begin{aligned} d:\ \Omega^1_N &\rightarrow \Omega^2_N, \\ \alpha &\mapsto [\Theta, \alpha]_+ - *_H(\alpha). \end{aligned} \tag{3-2}$$

where $*_H(\theta^a) = \frac{1}{2}\varepsilon^a_{bc}\theta^b\theta^c$. Finally, we can define a modified Hodge-star operator

$$\begin{aligned} *(\theta^a\theta^0) &= \frac{1}{2}\varepsilon^a_{bc}\theta^b\theta^c \\ *(\theta^a\theta^b) &= \varepsilon^{ab}_c\theta^c\theta^0, \end{aligned}$$

will play an essential role in the sequel. It is clearly a generalization of the definition in Section 2.3, but now maps indeed 2-forms into 2-forms. In particular,

$$*(\theta\theta^0) = \theta\theta. \tag{3-3}$$

No length scale has been introduced so far, only an integer N.

There are now different possibilities to introduce a scale into the theory, leading to different classical limits. The first is to consider the fuzzy sphere to have a fixed radius as in (2-27),

$$\varepsilon^{ij}_k x_i x_j = \Lambda_N\, x_k, \tag{3-4}$$

$$x \cdot x := g^{ij}x_i x_j = r^2 \tag{3-5}$$

where

$$\Lambda_N = r\,\frac{2}{\sqrt{N(N+2)}}. \tag{3-6}$$

Another possibility is to fix

$$\varepsilon_k^{ij} y_i y_j = \sqrt{\bar{k}}\, y_k, \tag{3-7}$$
$$y \cdot y = \frac{N(N+2)}{4}\bar{k}. \tag{3-8}$$

introducing a length scale $\sqrt{\bar{k}}$ [37]. Then the radius is basically $r = N/2\ \sqrt{\bar{k}}$ for large N. However, we will develop the formalism of gauge theories without introducing any scale here, and use λ_a which satisfy (3-1). Space then acquires an onion-like structure with an infinite sequence of concentric fuzzy spheres at the radii given by the above relation (i.e. $\bar{k} = 1$). This is also the scenario which is appropriate to study D-branes on $SU(2)$, where transitions between various such spheres are indeed expected. There are further possibilities, which were discussed in some detail in [37].

We define the integral over functions as

$$\int_{S^2_N} f = 4\pi\ \mathrm{tr}(f).$$

Similarly, we define the integral over a 3-form $\alpha = f dV$ where $dV = \theta\theta\theta$ by

$$\int_{S^2_N} \alpha = \int_{S^2_N} f. \tag{3-9}$$

This leads to a quantization condition on the area:

$$\text{Area}[S^2_N] = \int_{S^2_N} dV \quad \approx\ 4\pi\ N$$

Stokes theorem is easily seen to work for the present situation, noting that the sphere has no boundary. Therefore

$$\int_{S^2_N} d\alpha^{(2)} = 0$$

for any 2-form $\alpha^{(2)}$ on the fuzzy sphere, and

$$\int_{S^2_N \times \mathbb{R}} d\alpha^{(3)} = \int_{S^2_N} \alpha^{(3)}(t=\infty) - \int_{S^2_N} \alpha^{(3)}(t=-\infty)$$

for any 3-form $\alpha^{(3)}$ on $S^2_N \times \mathbb{R}$.

3.1.2 Gauge theory.

The definition of a gauge theory on the fuzzy sphere can be taken over from Section 2.4.2, setting $q = 1$ and adding the time variable. The degrees of freedom of a $U(1)$ gauge theory are encoded in a one-form $B = \sum B_\alpha \theta^\alpha$. It is naturally written as $B = \theta + A$ defining the "connection" 1-form $A = A_\alpha \theta^\alpha$. Hence $B_a = \lambda_a + A_a$ and $B_0 = A_0$. Notice that A_α is dimensionless here. The curvature is then defined formally as

$$F = (B + d_t)(B + d_t) - *_H(B) = (dA + A^2) = \frac{1}{2} F_{\alpha\beta}\theta^\alpha\theta^\beta,$$

cp. (2-124). Its components are simply

$$F_{0a} = \dot{B}_a + [\lambda_a, B_0] + [B_0, B_a], \qquad F_{ab} = [B_a, B_b] - \varepsilon^c{}_{ab} B_c.$$

The dynamics of the model is defined by the Yang-Mills action

$$\boxed{S_{YM} = \int_{S^2_N \times \mathbb{R}} F * F} \tag{3-10}$$

which is invariant under the gauge transformations

$$B \to U^{-1} B U, \quad A \to U^{-1} A U + U^{-1} dU \tag{3-11}$$

One can now easily derive equations of motion [37]. Moreover, we can conclude immediately from the inequality

$$\int_{S^2_N \times \mathbb{R}} (F \pm *F) * (F \pm *F) \geq 0$$

and

$$FF = dK = d(AdA + \frac{2}{3} A^3) \tag{3-12}$$

as well as

$$\int_{S^2_N \times \mathbb{R}} dK = \int_{S^2_N} K(t = \infty) - \int_{S^2_N} K(t = -\infty)$$

that the action $S[A]$ is bounded from below by

$$S_{YM} \geq \Big| \int_{S^2_N \times \mathbb{R}} FF \Big| = \Big| \int_{S^2_N} K(t = \infty) - \int_{S^2_N} K(t = -\infty) \Big|$$

for any configuration A "in the same sector" determined by the boundary values of $\int_{S^2_N} K(t = \pm\infty)$. This is exactly the same argument as in the undeformed case.

3.1.3 Zero curvature solutions and multi-brane configurations.

One particular solution of the zero-curvature condition $F = 0$ is given by $F = \theta$. More generally, its time-independent solutions have the form

$$B = \oplus_i \lambda_a^{(i)} \theta^a \tag{3-13}$$

as in Section 2.4.2, where $\lambda_a^{(i)}$ are irreducible representations of the Lie algebra $su(2)$ acting on n_i-dimensional subspaces of $\mathbb{C}^n$ where $n = N + 1$. The solutions to $F = 0$ are hence determined by the partitions $[n] = [n_1, ..., n_k]$ of n such that $\sum n_i = n$, which can be interpreted as "multi-brane" configuration consisting of branes (fuzzy spheres) $S^2_{n_i-1}$. A better interpretation of our gauge theory is therefore as describing n "$D0$-branes", which can form bound states as fuzzy spheres of various sizes. This fits nicely with the string-theoretical picture of $D0$-branes on group manifolds, in the limit of infinite radius of the group.

3.2 Fuzzy Instantons

From the above considerations, it follows that solutions of the self-duality conditions

$$\boxed{F = \pm * F} \tag{3-14}$$

are certainly solutions of the equations of motions, and moreover they are the solutions with the smallest action for any configuration A with given boundary values at $t = \pm\infty$. We shall now find these solutions, called instantons. First, it is useful to go to the Coulomb gauge $B_0 = 0$. We then make the Ansatz $B_a = f(t)\lambda_a$, which can be also written as

$$B = f(t)\theta, \qquad A = (f(t) - 1)\theta.$$

The $f(t)$ is *a priori* an arbitrary element of the algebra. The field strength is then

$$F = dA + A^2 = df\theta + (f - 1)\theta\theta + (f - 1)\theta(f - 1)\theta. \tag{3-15}$$

We shall compute only the simple case with $f(t)$ in the center. This is reasonable, because it describes the configurations which are invariant under $SU(2)$. In this case

$$F = \dot{f} dt\theta + (f - 1) f\theta\theta,$$

and self-duality condition becomes

$$\dot{f} dt\theta + (f - 1) f\theta^2 = \pm(\dot{f}\ (-\theta^2) - (f - 1) f dt\theta).$$

Hence

$$\dot{f} = \mp(f - 1) f. \tag{3-16}$$

The constant solutions $f = 0, 1$ correspond to the two stable ground states of the system, $B = 0$ and $B = \theta$. The instanton solution interpolates between the first two, and is given by

$$f_+ = \frac{1}{2}(1 + \tanh(\frac{t}{2} + b)), \qquad b \in \mathbb{R} \tag{3-17}$$

interpolates between $f(-\infty) = 0$ and $f(+\infty) = 1$. This has the same form as the classical double-well instanton [102]. The "anti-instanton" is given by

$$f_- = \frac{1}{2}(1 + \tanh(-\frac{t}{2} + b)), \qquad b \in \mathbb{R} \tag{3-18}$$

connecting the vacua $f(-\infty) = 1$ and $f(+\infty) = 0$.

To calculate the value of the action S_{YM} for this instanton, it is again convenient to use the fact that the 4-form F^2 is the exterior derivative of a 3-form,

$$\int_{S_N^2 \times \mathbb{R}} F^2 = \int_{S_N^2 \times \mathbb{R}} dK, \qquad K = AdA + \frac{2}{3}A^3.$$

The action then becomes

$$S = \int_{S_N^2} K(+\infty) - \int_{S_N^2} K(-\infty).$$

Our solution tunnels between $f = 0$ at $t = -\infty$ to $f = 1$ at $t = \infty$. That is $K(+\infty) = 0$ and

$$K(-\infty) = \frac{1}{3}\theta^3 = \frac{1}{3}\, dV.$$

But

$$\int_{S_N^2} \theta^3 = 4\pi(N+1)$$

up to an arbitrary overall normalization constant. We find then that the action of the instanton solution (3-17) is

$$S = -\frac{1}{3}\int_{S_N^2} \theta^3 = \frac{4\pi}{3}\,(N+1).$$

In particular, this is a lower bound for the action of any field configuration. The precise value is of course somewhat arbitrary, depending on the normalization of the integral. However, it acquires meaning in the case of multiple branes, which we will now discuss.

3.2.1 Multi-brane instantons and Fock space

We have found an instanton solution which tunnels between $f = 0$ and $f = 1$. If we return to the definition of the curvature as a functional of the fields A, we see that this corresponds to a transition from the irreducible representation of dimension $n = N + 1$ to the completely reducible representation of the same dimension. The former corresponds to the partition $[n]$ of

n; the latter to the partition $[1,\dots 1]$. Now recall that the general solution of $F=0$ is given by multi-brane configurations labeled by partitions $[n_1,...,n_k]$. By the same construction there is, for each index i, an instanton which tunnels between $[n_i]$ and $[1,\dots 1]$. There are perhaps other instantons, those which correspond to transitions between non-trivial projectors.

Consider for example an instanton T_l^n which tunnels as

$$[l, \underbrace{1,\cdots 1}_{n-l}] \xrightarrow{T_l^n} [\underbrace{1,\cdots 1}_{n}].$$

The corresponding gauge field is $B_a = f(t)e_l\ \lambda_a$ where e_l is the projector on $\mathbb{C}^l$. Using the same definitions of curvature and integral as before, the only thing that changes in the above calculation[1] is that now

$$\int_{S_N^2} e_l dV = 4\pi l$$

and the action is given by

$$\boxed{S[T_l^n] = -\frac{\pi}{3}\int_{S_N^2} e_l\theta^3 = \frac{4\pi}{3}\, l.} \tag{3-19}$$

This is the basic relation which we find. The action of such instantons is basically an integer l, or the sum $\sum l_i$ for multi-instantons of sizes l_i interpolating between the corresponding branes. This is quite remarkable: of course classically, the action of instantons is always an integer, given by the Chern number; the reason is that F is a connection on a nontrivial bundle. The physical and geometric meaning of the present situation is very different and entirely non-classical. Nevertheless, the integral over the corresponding action is still basically integer-valued, and measures certain "topological" classes.

Fock space. Consider a general partition $[n_1,\cdots,n_k]$. Let e_i be the projector onto the ith sector. An arbitrary projector e can be written in the form

$$e = \sum_i \epsilon_i e_i, \qquad \epsilon_i = 0,1.$$

The corresponding expression (3-15) for the field strength is given by

$$F = \sum_i F_i$$

with each F_i of the form (3-15). Each corresponding f_i evolves independently according to its field equation. The instantons tunnel therefore between different partitions.

If one quantizes the system one obtains a bosonic Fock space of ordinary 'vacuum modes'. Due to the gauge symmetry, different orderings of the partition $[n_1,\cdots,n_k]$ are equivalent, and

[1] note in particular that the frame θ^a is independent of l

we can describe the vacuum states (branes) as a bosonic Fock space labeled by the multiplicities of each brane,

$$|k_1, k_2, \cdots, k_n\rangle = [\underbrace{1, \cdots, 1}_{k_1}, \underbrace{2, \cdots, 2}_{k_2}, ..., \underbrace{n, \cdots, n}_{k_n}].$$

The integer k_j is the occupation number of the brane $[j]$, which is a fuzzy sphere S^2_{j-1}. Besides these vacuum modes there is also a Fock space $\mathcal{F}$ of 'tunneling modes', which transform a brane $[j]$ into j branes [1]. In Fock-space notation the basic transition is of the general form

$$|k_1, \cdots, k_i, \cdots, k_n\rangle \xleftrightarrow{T^n_i} |k_1 + i, \cdots, k_i - 1, \cdots, k_n\rangle.$$

One of the k_i irreducible components of dimension i in the given representation 'decays' into i representations of dimension one, or the inverse. The tunneling modes interact with all the vacuum modes and change their energy eigenvalues in a rather complicated way. Without the tunneling modes the vacuum modes do not interact, so we consider the tunneling modes as responsible for the dynamics. An instanton gas is an ensemble of tunneling modes, considered as a Bose gas.

In physical terms, instantons describe quantum mechanical tunneling between these brane configurations; see e.g. [103] for a lucid discussion of the physical aspects of instantons. One could therefore try to use them for calculating transition rates between such configurations. The probability of these transitions is proportional to the barrier penetration rate [104]

$$p[T^n_j] = A[T^n_j]e^{-S[T^n_j]}.$$

The $A[T^n_i]$ is a WKB amplitude difficult to calculate in general. Furthermore, while all the different partitions are degenerate classically, the tunneling phenomena would lift this degeneracy and make some partitions more favorable. Consider the case $n = 2$ with its two partitions $[2]$ and $[1, 1]$. The two degenerate levels split by an amount proportional to the transition probability $p = Ae^{-8\pi/3}$. Such processes are indeed predicted by string theory.

To discuss this situation in more detail, one should consider an instanton gas consisting of many bounces, which is to some extent discussed in [37]. This is a complicated problem, and we will not pursue this here any further. One can see the complexity of the situation also in a different way, noting that the degeneracy of the vacua is lifted by the zero-point fluctuations. For example, the completely reducible configuration $[1, ..., 1]$ and the irreducible configuration $[n]$ will have vacuum energy given respectively by

$$E_{[1,\ldots,1]} \simeq \frac{1}{2}\hbar\omega \cdot n^2, \qquad E_{[n]} \simeq \frac{1}{2}\hbar\omega \cdot \sum 2l(2l+1), \qquad \omega \approx 1.$$

In the first there are n^2 modes of equal frequency ω, because there is no kinetic term. In the second, for each $l \lesssim n$ there are $2(2l + 1)$ modes of frequency $l\omega$, due to the kinetic energy of the gauge fields. The remaining vacuum energies lie between these two values.

Chapter 4

One–loop effects on the fuzzy sphere

This chapter covers the first one-loop calculation of quantum effects on the fuzzy sphere, which was done in a joint work [32] with John Madore and Chong-Sun Chu. We consider scalar Φ^4 theory, and calculate the two point function at one loop. The fuzzy sphere S^2_N is characterized by its radius R and a "noncommutativity" parameter N which is an integer. It approaches the classical sphere in the limit $N \to \infty$ for fixed R, and can be thought of as consisting of N "quantum cells". The algebra of functions on S^2_N is finite, with maximal angular momentum N. Nevertheless, it admits the full symmetry group $SO(3)$ of motions. The fuzzy sphere is closely related to several other noncommutative spaces [36, 38]. In particular, it can be used as an approximation to the quantum plane $\mathbb{R}^2_\theta$, by "blowing up" for example the neighborhood of the south pole. This is the quantum spaces with the basic commutation relations

$$[x_i, x_j] = i\theta_{ij}$$

for a constant antisymmetric tensor θ_{ij} which has been studied extensively in recent years. Many of the problems that can arise in QFT on noncommutative spaces are illustrated in this much–studied example of the noncommutative plane $\mathbb{R}^n_\theta$; see [44] for a recent review. One of the most intriguing phenomena on that space is the existence of an ultraviolet/infrared (UV/IR) mixing [105] in the quantum effective action. Due to this mixing, an IR singularity arises from integrating out the UV degrees of freedom. This threatens the renormalizability and even the existence of a QFT. Hence a better understanding (beyond the technical level) of the mechanism of UV/IR mixing and possible ways to resolve it are certainly highly desirable. One possible approach is to approximate $\mathbb{R}^n_\theta$ in terms of a different noncommutative space. This idea is realized here, approximating $\mathbb{R}^2_\theta$ by a fuzzy sphere. This will allow to understand the UV/IR mixing as an infinite variant of a "noncommutative anomaly" on the fuzzy sphere, which is a closely related but different phenomenon discussed below. This is one of our main results. A related, but less geometric approach was considered in [106].

The fuzzy sphere has the great merit that it is very clear how to quantize field theory on it, using a finite analog of the path integral [61]. Therefore QFT on this space is a priori completely well–defined, on a mathematical level. Nevertheless, it is not clear at all whether such a theory

makes sense from a physical point of view, i.e. whether there exists a limiting theory for large N, which could be interpreted as a QFT on the classical sphere. There might be a similar UV/IR problem as on the quantum plane $\mathbb{R}^2_\theta$, as was claimed in a recent paper [107]. In other words, it is not clear if and in what sense such a QFT is renormalizable. As a first step, we calculate the two point function at one loop and find that it is well defined and regular, without UV/IR mixing. Moreover, we find a closed formula for the two point function in the commutative limit, i.e. we calculate the leading term in a $1/N$ expansion.

It turns out that the 1–loop effective action on S^2_N in the commutative limit differs from the 1–loop effective action on the commutative sphere S^2 by a finite term, which we call "noncommutative anomaly" (NCA). It is a mildly nonlocal, "small" correction to the kinetic energy on S^2, and changes the dispersion relation. It arises from the nonplanar loop integration. Finally, we consider the planar limit of the fuzzy sphere. We find that a IR singularity is developed in the nonplanar two point function, and hence the UV/IR mixing emerges in this limit. This provides an understanding of the UV/IR mixing for QFT on $\mathbb{R}^2_\theta$ as a "noncommutative anomaly" which becomes singular in the planar limit of the fuzzy sphere dynamics.

This chapter is organized as follows. In Section 4.1, we consider different geometrical limits of the classical (ie. $\hbar = 0$) fuzzy sphere. In particular, we show how the commutative sphere and the noncommutative plane $\mathbb{R}^2_\theta$ can be obtained in different corners of the moduli space of the fuzzy sphere. In Section 4.2, we study the quantum effects of scalar Φ^4 field theory on the fuzzy sphere at 1-loop. We show that the planar and nonplanar 2-point function are both regular in the external angular momentum and no IR singularity is developed. This means that there is no UV/IR mixing phenomenon on the fuzzy sphere. We also find that the planar and nonplanar two point functions differ by a finite amount which is smooth in the external angular momentum, and survives in the commutative limit. Therefore the commutative limit of the Φ^4 theory at one loop differs from the corresponding one loop quantum theory on the commutative sphere by a finite term (4-38). In section 4, we consider the planar limit of this QFT, and recover the UV/IR mixing.

4.1 More on the Fuzzy Sphere and its Limits

4.1.1 The multiplication table for S^2_N

We need some more explicit formulas for the multiplication on S^2_N. Recall from Section 2.1 that the algebra of functions on the fuzzy sphere is generated by Hermitian operators $x = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\lambda_N \eta_{ijk} x_k, \quad x_1^2 + x_2^2 + x_3^2 = R^2. \tag{4-1}$$

The radius R is quantized in units of λ_N by (2-4)

$$\frac{R}{\lambda_N} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}, \quad N = 1, 2, \cdots \tag{4-2}$$

The algebra of "functions" S^2_N is simply the algebra $Mat(N+1)$ of $(N+1)\times(N+1)$ matrices. It is covariant under the adjoint action of $SU(2)$, under which it decomposes into the irreducible representations with dimensions $(1)\oplus(3)\oplus(5)\oplus ... \oplus(2N+1)$. The integral of a function $F\in S^2_N$ over the fuzzy sphere is given by

$$R^2\int F = \frac{4\pi R^2}{N+1}\mathrm{tr}[F(x)], \tag{4-3}$$

where we have introduced $\int$, the integral over the fuzzy sphere with unit radius. It agrees with the integral $\int d\Omega$ on S^2 in the large N limit. One can also introduce the inner product

$$(F_1,F_2) = \int F_1^\dagger F_2. \tag{4-4}$$

A complete basis of functions on S^2_N is given by the $(N+1)^2$ spherical harmonics, Y^J_j, $(J = 0,1,...,N; -J\le j\le J)$ [1], which are the weight basis of the spin J component of S^2_N explained above. They correspond to the usual spherical harmonics, however the angular momentum has an upper bound N here. This is a characteristic feature of fuzzy sphere. The normalization and reality for these matrices can be taken to be

$$(Y^J_j, Y^{J'}_{j'}) = \delta_{JJ'}\delta_{jj'}, \qquad (Y^J_j)^\dagger = (-1)^J Y^J_{-j}. \tag{4-5}$$

They obey the "fusion" algebra

$$\begin{aligned} Y^I_i Y^J_j \;&=\; \sqrt{\frac{N+1}{4\pi}}\sum_{K,k}(-1)^{2\alpha+I+J+K+k}\sqrt{(2I+1)(2J+1)(2K+1)}\,\cdot \\ &\cdot\begin{pmatrix} I & J & K\\ i & j & -k\end{pmatrix}\begin{Bmatrix} I & J & K\\ \alpha & \alpha & \alpha\end{Bmatrix} Y^K_k, \end{aligned} \tag{4-6}$$

where the sum is over $0\le K\le N$, $-K\le k\le K$, and

$$\alpha = N/2. \tag{4-7}$$

Here the first bracket is the Wigner $3j$-symbol and the curly bracket is the $6j$–symbol of $su(2)$, in the standard mathematical normalization [108]. Using the Biedenharn–Elliott identity (4-55), it is easy to show that (4-6) is associative. In particular, $Y^0_0 = \frac{1}{\sqrt{4\pi}}\,\mathbf{1}$. The relation (4-6) is independent of the radius R, but depends on the deformation parameter N. It is a deformation of the algebra of product of the spherical harmonics on the usual sphere. We will need (4-6) to derive the form of the propagator and vertices in the angular momentum basis.

Now we turn to various limits of the fuzzy sphere. By tuning the parameters R and N, one can obtain different limiting algebras of functions. In particular, we consider the commutative sphere S^2 and the noncommutative plane $\mathbb{R}^2_\theta$.

[1] We will use capital and small letter (e.g. (J,j)) to refer to the eigenvalue of the angular momentum operator $\mathbf{J^2}$ and J_z respectively.

4.1.2 The limit of the commutative sphere S^2

The commutative limit is defined by

$$N \to \infty, \quad \text{keeping } R \text{ fixed.} \tag{4-8}$$

In this limit, (2-2) reduces to $[x_i, x_j] = 0$ and we obtain the commutative algebra of functions on the usual sphere S^2. Note that (4-6) reduces to the standard product of spherical harmonics, due to the asymptotic relation between the $6j$–symbol and the Wigner $3j$-symbol [108],

$$\lim_{\alpha\to\infty}(-1)^{2\alpha}\sqrt{2\alpha}\left\{\begin{array}{ccc} I & J & K \\ \alpha & \alpha & \alpha \end{array}\right\} = (-1)^{I+J+K}\left(\begin{array}{ccc} I & J & K \\ 0 & 0 & 0 \end{array}\right). \tag{4-9}$$

4.1.3 The limit of the quantum plane $\mathbf{R}^2_\theta$

If the fuzzy sphere is blown up around a given point, it becomes an approximation of the quantum plane [29]. To obtain this planar limit, it is convenient to first introduce an alternative representation of the fuzzy sphere in terms of stereographic projection. Consider the generators

$$y_+ = 2Rx_+(R - x_3)^{-1}, \quad y_- = 2R(R - x_3)^{-1}x_-, \tag{4-10}$$

where $x_\pm = x_1 \pm ix_2$. The generators $y_\pm$ are the coordinates of the stereographic projection from the north pole. $y = 0$ corresponds to the south pole. Now we take the large N and large R limit, such that

$$N \to \infty, \quad R^2 = N\theta/2 \to \infty, \quad \text{keeping } \theta \text{ fixed.} \tag{4-11}$$

In this limit,

$$\frac{\lambda_N}{\sqrt{\theta}} \sim \frac{1}{\sqrt{N}} \tag{4-12}$$

and $[y_+, y_-] = -4R^2\lambda_N(R - x_3)^{-1} + o(\lambda_N^2)$. Since $y_+y_- = 4R^2(R + x_3)(R - x_3)^{-1} + o(N^{-1/2})$, we can cover the whole y-plane with $x_3 = -R + \beta/R$ with finite but arbitrary β. The commutation relation of the y generators takes the form

$$[y_+, y_-] = -2\theta \tag{4-13}$$

up to corrections of order λ_N^2, or

$$[y_1, y_2] = -i\theta \tag{4-14}$$

with $y_\pm = y_1 \pm iy_2$.

4.2 One Loop Dynamics of Φ^4 on the Fuzzy Sphere

Consider a scalar Φ^4 theory on the fuzzy sphere, with action

$$S_0 = \int \frac{1}{2}\,\Phi(\Delta + \mu^2)\Phi + \frac{g}{4!}\Phi^4. \tag{4-15}$$

Here Φ is Hermitian, μ^2 is the dimensionless mass square, g is a dimensionless coupling and $\Delta = \sum J_i^2$ is the Laplace operator. The differential operator J_i acts on function $F \in S_N^2$ as

$$J_i F = \frac{1}{\lambda_N}[x_i, F]. \tag{4-16}$$

This action is valid for any radius R, since μ and g are dimensionless. To quantize the theory, we will follow the path integral quantization procedure as explained in [61]. We expand Φ in terms of the modes,

$$\Phi = \sum_{L,l} a_l^L Y_l^L, \quad a_l^{L\dagger} = (-1)^l a_{-l}^L. \tag{4-17}$$

The Fourier coefficient a_l^L are then treated as the dynamical variables, and the path integral quantization is defined by integrating over all possible configuration of a_l^L. Correlation functions are computed using [61]

$$\langle a_{l_1}^{L_1} \cdots a_{l_k}^{L_k} \rangle = \frac{\int [\mathcal{D}\Phi] e^{-S_0}\; a_{l_1}^{L_1} \cdots a_{l_k}^{L_k}}{\int [\mathcal{D}\Phi] e^{-S_0}}. \tag{4-18}$$

For example, the propagator is

$$\langle a_l^L a_{l'}^{L'\dagger} \rangle = (-1)^l \langle a_l^L a_{-l'}^{L'} \rangle = \delta_{LL'} \delta_{ll'} \frac{1}{L(L+1)+\mu^2}, \tag{4-19}$$

and the vertices for the Φ^4 theory are given by

$$a_{l_1}^{L_1} \cdots a_{l_4}^{L_4}\; V(L_1, l_1; \cdots; L_4, l_4) \tag{4-20}$$

where

$$\begin{aligned} V(L_1, l_1; \cdots; L_4, l_4) \quad &= \frac{g}{4!}\frac{N+1}{4\pi}(-1)^{L_1+L_2+L_3+L_4} \prod_{i=1}^{4} (2L_i+1)^{1/2} \sum_{L,l} (-1)^l (2L+1) \cdot \\ &\cdot \begin{pmatrix} L_1 & L_2 & L \\ l_1 & l_2 & l \end{pmatrix} \begin{pmatrix} L_3 & L_4 & L \\ l_3 & l_4 & -l \end{pmatrix} \begin{Bmatrix} L_1 & L_2 & L \\ \alpha & \alpha & \alpha \end{Bmatrix} \begin{Bmatrix} L_3 & L_4 & L \\ \alpha & \alpha & \alpha \end{Bmatrix}. \end{aligned} \tag{4-21}$$

One can show that V is symmetric with respect to cyclic permutation of its arguments (L_i, l_i).

The $1PI$ two point function at one loop is obtained by contracting 2 legs in (4-21) using the propagator (4-19). The planar contribution is defined by contracting neighboring legs:

$$(\Gamma_{planar}^{(2)})_{ll'}^{LL'} = \frac{g}{4\pi}\frac{1}{3}\, \delta_{LL'} \delta_{l,-l'} (-1)^l \cdot I^P, \quad I^P := \sum_{J=0}^{N} \frac{2J+1}{J(J+1)+\mu^2}. \tag{4-22}$$

All 8 contributions are identical. Similarly by contracting non–neighboring legs, we find the non–planar contribution

$$
\begin{aligned}
(\Gamma^{(2)}_{nonplanar})^{LL'}_{ll'} &= \frac{g}{4\pi}\frac{1}{6}\,\delta_{LL'}\delta_{l,-l'}(-1)^l\cdot I^{NP},\\
I^{NP} &:= \sum_{J=0}^{N}(-1)^{L+J+2\alpha}\,\frac{(2J+1)(2\alpha+1)}{J(J+1)+\mu^2}\left\{\begin{array}{ccc}\alpha & \alpha & L\\ \alpha & \alpha & J\end{array}\right\}.
\end{aligned}
\tag{4-23}
$$

Again the 4 possible contractions agree. These results can be found using standard identities for the $3j$ and $6j$ symbols, see e.g. [108] and Section 4.5.

It is instructive to note that I^{NP} can be written in the form

$$
I^{NP} = \sum_{J=0}^{N}\frac{2J+1}{J(J+1)+\mu^2}\,f_J,
\tag{4-24}
$$

where f_J is obtained from the generating function

$$
f(x) = \sum_{J=0}^{\infty} f_J x^J = \frac{1}{1-x}\,{}_2F_1(-L,L+1,2\alpha+2,\frac{x}{x-1})\,{}_2F_1(-L,L+1,-2\alpha,\frac{x}{x-1}).
\tag{4-25}
$$

Here the hypergeometric function ${}_2F_1(-L,L+1;c;z)$ is a polynomial of degree L for any c. For example, for $L=0$, one obtains

$$
f_J = 1, \quad 0\leq J\leq N,
\tag{4-26}
$$

and hence the planar and nonplanar two point functions coincides. For $L=1$, we have

$$
f_J = 1-\frac{J(J+1)}{2\alpha(\alpha+1)}, \quad 0\leq J\leq N\ ,
\tag{4-27}
$$

and hence

$$
I^{NP} = I^P - \frac{1}{2\alpha(\alpha+1)}\sum_{J=0}^{2\alpha}\frac{J(J+1)(2J+1)}{J(J+1)+\mu^2}.
\tag{4-28}
$$

Note that the difference between the planar and nonplanar two point functions is finite. It is easy to convince oneself that for any finite external angular momentum L, the difference between the planar and nonplanar two point function is finite and analytic in $1/\alpha$. This fact is important as it implies that, unlike in the $\mathbb{R}^n_\theta$ case, there is no infrared singularity developed in the nonplanar amplitude[2]. We will have more to say about this later.

[2] It was argued in [107] that the nonplanar two-point function has a different sign for even and odd external angular momentum L. This is misleading, however, because only $J=2\alpha$ was considered there. This is not significant in the loop integral which is a sum over all J.

4.2.1 On UV/IR mixing and the commutative limit

Let us recall that in the case of noncommutative space $\mathbb{R}^n_\theta$, the one–loop contribution to the effective action often develops a singularity at $\theta p = 0$ [105, 109, 110, 111]. This infrared singularity is generated by integrating out the infinite number of degree of freedom in the nonplanar loop. This phenomenon is referred to as "UV/IR mixing", and it implies in particular that (1) the nonplanar amplitude is singular when the external momentum is zero in the noncommutative directions; and (2) the quantum effective action in the commutative limit is different from the quantum effective action of the commutative limit [112].

- Effective action on the fuzzy sphere.

We want to understand the behavior of the corresponding planar and nonplanar two point functions on the fuzzy sphere, to see if there is a similar UV/IR phenomenon. We emphasize that this is not obvious a priori even though quantum field theory on the fuzzy sphere is always finite. The question is whether the 2–point function is smooth at small values of L, or rapidly oscillating as was indeed claimed in a recent paper [107]. Integrating out all the degrees of freedom in the loop could in principle generate a IR singularity, for large N.

However, this is not the case. We found above that the planar and nonplanar two point function agree precisely with each other when the external angular momentum $L = 0$. For general L, a closed expression for g_J for general L is difficult to obtain. We will derive below an approximate formula for the difference $I^{NP} - I^P$, which is found to be an excellent approximation for large N by numerical tests, and becomes exact in the commutative limit $N \to \infty$.

First, the planar contribution to the two point function

$$I^P = \sum_{J=0}^{N} \frac{2J+1}{J(J+1)+\mu^2} \tag{4-29}$$

agrees precisely with the corresponding terms on the classical sphere as $N \to \infty$, and it diverges logarithmically

$$I^P \sim \log\alpha + o(1). \tag{4-30}$$

To understand the nonplanar contribution, we start with the following approximation formula [108] for the $6j$ symbols due to Racah,

$$\left\{ \begin{array}{ccc} \alpha & \alpha & L \\ \alpha & \alpha & J \end{array} \right\} \approx \frac{(-1)^{L+2\alpha+J}}{2\alpha} P_L(1 - \frac{J^2}{2\alpha^2}), \tag{4-31}$$

where P_L are the Legendre Polynomials. This turns out to be an excellent approximation for all $0 \leq J \leq 2\alpha$, provided α is large and $L \ll \alpha$. Since this range of validity of this approximation formula is crucial for us, we shall derive it in Section 4.5. This allows then to rewrite the sum in (4-23) to a very good approximation as

$$I^{NP} - I^P = \sum_{J=0}^{2\alpha} \frac{2J+1}{J(J+1)+\mu^2} \left(P_L(1 - \frac{J^2}{2\alpha^2}) - 1 \right) \tag{4-32}$$

for large α. Since $P_L(1) = 1$ for all L, only $J \gg 1$ contributes, and one can approximate the sum by the integral

$$\begin{aligned} I^{NP} - I^P \qquad &\approx \int_0^2 du \, \frac{2u + \frac{1}{\alpha}}{u^2 + \frac{u}{\alpha} + \frac{\mu^2}{\alpha^2}} \left(P_L(1 - \frac{u^2}{2}) - 1 \right) \\ &= \int_{-1}^{1} dt \, \frac{1}{1-t}(P_L(t) - 1) + o(1/\alpha), \end{aligned} \tag{4-33}$$

assuming $\mu \ll \alpha$. This integral is finite for all L. Indeed using generating functions techniques, it is easy to show that

$$\int_{-1}^{1} dt \frac{1}{1-t}(P_L(t) - 1) = -2\,(\sum_{k=1}^{L} \frac{1}{k}) = -2h(L), \tag{4-34}$$

where $h(L) = \sum_{k=1}^{L} \frac{1}{k}$ is the harmonic number and $h(0) = 0$. While $h(L) \approx logL$ for large L, it is finite and well–behaved for small L. Therefore we obtain the effective action

$$\boxed{S_{one-loop} = S_0 + \int \tfrac{1}{2}\, \Phi(\delta\mu^2 - \tfrac{g}{12\pi} h(\widetilde{\Delta}))\Phi + o(1/\alpha)} \tag{4-35}$$

to the first order in the coupling where

$$\delta\mu^2 = \frac{g}{8\pi} \sum_{J=0}^{N} \frac{2J+1}{J(J+1) + \mu^2} \tag{4-36}$$

is the mass square renormalization, and $\widetilde{\Delta}$ is the function of the Laplacian which has eigenvalues L on Y_l^L. Thus we find that the effects due to noncommutativity are analytic in the noncommutative parameter $1/\alpha$. This is a finite quantum effect with nontrivial, but mild L dependence. Therefore no IR singularity is developed, and we conclude that there is no UV/IR problem on the fuzzy sphere[3].

- The commutative limit

The commutative limit of the QFT is defined by the limit

$$\alpha \to \infty, \quad \text{keeping } R,\, g,\, \mu \text{ fixed.} \tag{4-37}$$

In this limit, the resulting one-loop effective action differs from the effective action obtained by quantization on the commutative sphere by an amount

$$\boxed{\Gamma^{(2)}_{NCA} = -\tfrac{g}{24\pi} \int \Phi h(\widetilde{\Delta})\Phi.} \tag{4-38}$$

[3]The author of [107] argued that the effective action is not a smooth function of the external momentum and suggested this to be a signature of UV/IR mixing. We disagree with his result.

We refer to this as a "NonCommutative Anomaly", since it is the piece of the quantum effective action which is slightly nonlocal and therefore not present in the classical action. "Noncommutative" also refers to fact that the quantum effective action depends on whether we quantize first or take the commutative limit first.

The new term $\Gamma^{(2)}_{NCA}$ modifies the dispersion relation on the fuzzy sphere. It is very remarkable that such a "signature" of an underlying noncommutative space exists, even as the noncommutativity on the geometrical level is sent to zero. A similar phenomena is the induced Chern-Simon term in 3-dimensional gauge theory on $\mathbb{R}^3_\theta$ [112]. This has important implications on the detectability of an underlying noncommutative structure. The reason is that the vacuum fluctuations "probe" the structure of the space even in the UV, and depend nontrivially on the external momentum in the nonplanar diagrams. Higher–order corrections may modify the result. However since the theory is completely well–defined for finite N, the above result (4-38) is meaningful for small coupling g.

Summarizing, we find that quantization and taking the commutative limit does not commute on the fuzzy sphere, a fact which we refer to as "noncommutative anomaly". A similar phenomenon also occurs on the noncommutative quantum plane $\mathbb{R}^n_\theta$. However, in contrast to the case of the quantum plane, the "noncommutative anomaly" here is not due to UV/IR mixing since there is no UV/IR mixing on the fuzzy sphere. We therefore suggest that the existence of a "noncommutative anomaly" is a generic phenomenon and is independent of UV/IR mixing[4]. One can expect that the "noncommutative anomaly" does not occur for supersymmetric theories on the 2–sphere.

4.3 Planar Limit of Quantum Φ^4

In this section, we consider the planar limit of the Φ^4 theory on the fuzzy sphere at one loop. Since we have shown that there is no UV/IR mixing on the fuzzy sphere, one may wonder whether (4-47) could provide a regularization for the nonplanar two point function (4-48) on $\mathbb{R}^2_\theta$ which does not display an infrared singularity. This would be very nice, as this would mean that UV/IR can be understood as an artifact that arises out of a bad choice of variables. However, this is not the case.

To take the planar limit, we need in addition to (4-11), also

$$\mu^2 = m^2 R^2 \sim \alpha \to \infty, \quad \text{keeping } m \text{ fixed}, \tag{4-39}$$

so that a massive scalar theory is obtained. We wish to identify in the limit of large R the modes on the sphere with angular momentum L with modes on the plane with linear momentum p. This can be achieved by matching the Laplacian on the plane with that on the sphere in the large radius limit, ie.

$$L(L+1)/R^2 = p^2. \tag{4-40}$$

[4]However, as we will see, they are closely related.

It follows that

$$p = \frac{L}{R}. \tag{4-41}$$

Note that by (4-11), a mode with a fixed nonzero p corresponds to a mode on the sphere with large L:

$$L \sim R \sim \sqrt{\alpha}. \tag{4-42}$$

Since L is bounded by α, there is a UV cutoff Λ on the plane at

$$\Lambda = \frac{2\alpha}{R}. \tag{4-43}$$

Denote the external momentum of the two point function by p. It then follows that $\alpha \gg L \gg 1$ as long as $p \neq 0$.

It is easy to see that the planar amplitude (4-22) becomes

$$I^P = 2\int_0^\Lambda dk \frac{k}{k^2+m^2} \tag{4-44}$$

in the quantum plane limit, with $k = J/R$. This is precisely the planar contribution to the two point function on $\mathbb{R}^2_\theta$.

For the nonplanar two point function (4-23), we can again use the formula (4-31) which is valid for all J and large α, since the condition $\alpha \gg L$ is guaranteed by (4-42). Therefore

$$I^{NP}(p) = 2\int_0^\Lambda dk\, \frac{k}{k^2+m^2} P_{pR}(1-2\frac{k^2}{\Lambda^2}) \tag{4-45}$$

For large $L = pR$, we can use the approximation formula

$$P_L(\cos\phi) = \sqrt{\frac{\phi}{\sin\phi}} J_0((L+1/2)\phi) \quad + \; O(L^{-3/2}), \tag{4-46}$$

which is uniformly convergent [113] as $L \to \infty$ in the interval $0 \leq \phi \leq \pi - \epsilon$ for any small, but finite $\epsilon > 0$. Then one obtains

$$\begin{aligned} I^{NP}(p) &\approx 2\int_0^\Lambda dk\, \frac{k}{k^2+m^2}\sqrt{\frac{\phi_k}{\sin\phi_k}}\; J_0(pR\phi_k) \\ &\approx 2\int_0^\Lambda dk\, \frac{k}{k^2+m^2}\; J_0(\theta p k), \end{aligned} \tag{4-47}$$

where $\phi_k = 2\arcsin(k/\Lambda)$. The singularity at $\phi = \pi$ on the rhs of (4-46) (which is an artefact of the approximation and not present in the lhs) is integrable and does not contribute to (4-47) for large $p\Lambda\theta$. The integrals in (4-47) are (conditionally) convergent for $p \neq 0$, and the

approximations become exact for $p\Lambda\theta \to \infty$. Therefore we recover precisely the same form as the one loop nonplanar two point function on $\mathbb{R}^2_\theta$,

$$\frac{1}{2\pi}\int_0^\Lambda d^2k \, \frac{1}{k^2+m^2} e^{i\theta p \times k}. \tag{4-48}$$

For small $p\Lambda\theta$, i.e. in the vicinity of the induced infrared divergence on $\mathbb{R}^2_\theta$, these approximations are less reliable. We can obtain the exact form of the infrared divergence from (4-34),

$$I^{NP} = -2\log(p\sqrt{\theta}) + (I^P - \log\alpha). \tag{4-49}$$

Hence we find the same logarithmic singularity in the infrared as on $\mathbb{R}^2_\theta$ [105]. In other words, we find that the UV/IR mixing phenomenon which occurs in QFT on $\mathbb{R}^2_\theta$ can be understood as the infinite limit of the noncomutative anomaly (4-38) on the fuzzy sphere. Hence one could use the fuzzy sphere as a regularization of $\mathbb{R}^2_\theta$, where the logarithmic singularity $\log(p\sqrt{\theta})$ gets "regularized" by (4-34).

4.4 Discussion

We have done a careful analysis of the one–loop dynamics of scalar Φ^4 theory on the fuzzy sphere S^2_N. It turns out that the two point function is completely regular, without any UV/IR mixing problem. We also give a closed expression for the two point function in the commutative limit, i.e. we find an exact form for the leading term in a $1/N$ expansion. Using this we discover a "noncommutative anomaly" (NCA), which characterizes the difference between the quantum effective action on the commutative sphere S^2 and the commutative limit $N \to \infty$ of the quantum effective action on the fuzzy sphere. This anomaly is finite but mildly nonlocal on S^2, and changes the dispersion relation. It arises from the nonplanar loop integration.

It is certainly intriguing and perhaps disturbing that even an "infinitesimal" quantum structure of (space)time has a finite, nonvanishing effect on the quantum theory. Of course this was already found in the UV/IR phenomenon on $\mathbb{R}^n_\theta$, however in that case one might question whether the quantization procedure based on deformation quantization is appropriate. On the fuzzy sphere, the result is completely well-defined and unambiguous. One might argue that a "reasonable" QFT should be free of such a NCA, so that the effective, macroscopic theory is insensitive to small variations of the structure of spacetime at short distances. On the other hand, it is conceivable that our world is actually noncommutative, and the noncommutative dynamics should be taken seriously. Then there is no reason to exclude theories with NCA. In particular, one would like to understand better how sensitive these "noncommutative anomalies" are to the detailed quantum structure of spacetime.

By approximating the QFT on $\mathbb{R}^2_\theta$ with the QFT on the fuzzy sphere, we can explain the UV/IR mixing from the point of view of the fuzzy sphere as a infinite variant of the NCA. In some sense, we have regularized $\mathbb{R}^2_\theta$. It would be interesting to provide an explanation of the

UV/IR mixing also for the higher dimensional case $\mathbb{R}^4_\theta$. To do this, the first step is to realize $\mathbb{R}^4_\theta$ as a limit of a "nicer" noncommutative manifold. A first candidate is the product of two fuzzy spheres. Much work remains to be done to clarify this situation.

It would also be very desirable to include fermions and gauge fields in these considerations. In particular it will be interesting to determine the dispersion relation for "photons", depending on the "fuzzyness" of the underlying geometry. In the case of noncommutative QED on $\mathbb{R}^4_\theta$, this question was studied in [109, 110, 111] where a nontrivial modification to the dispersion relation of the "photon" was found which makes the theory ill–defined. In view of our results, one may hope that these modifications are milder on the fuzzy sphere and remain physically sensible.

4.5 Technical supplements

We derive the approximation formula (4-31) for large α and $0 \leq J \leq 2\alpha$, assuming $L \ll \alpha$.

There is an exact formula for the $6j$ coefficients due to Racah (see e.g. [108]), which can be written in the form

$$\begin{aligned} \left\{ \begin{array}{ccc} \alpha & \alpha & L \\ \alpha & \alpha & J \end{array} \right\} &= (-1)^{2\alpha+J} \sum_n (-1)^n \cdot \\ &\cdot \left(\begin{array}{c} L \\ n \end{array} \right)^2 \frac{(2\alpha-L)!(2\alpha+J+n+1)!(2\alpha-J)!(J!)^2}{(2\alpha+L+1)!(2\alpha+J+1)!(2\alpha-J-n)!((J-L+n)!)^2}. \end{aligned}$$

The sum is from $n = \max\{0, L-J\}$ to $\min\{L, 2\alpha-J\}$, so that all factorials are non-negative. Assume first that $L \leq J \leq 2\alpha - L$, so that the sum is from 0 to L. Since $\alpha \gg L$, this becomes

$$\left\{ \begin{array}{ccc} \alpha & \alpha & L \\ \alpha & \alpha & J \end{array} \right\} \approx (-1)^{2\alpha+J} \frac{1}{(2\alpha)^{2L+1}} \sum_{n=0}^{L} (-1)^n \left(\begin{array}{c} L \\ n \end{array} \right)^2 (4\alpha^2 - J^2)^n \left(\frac{J!}{(J-(L-n))!} \right)^2, \tag{4-50}$$

dropping corrections of order $o(\frac{L}{\alpha})$. Now there are 2 cases: either $J \gg L$, or otherwise $J \ll \alpha$ since $\alpha \gg L$. Consider first

1. $J \gg L$:

   Then $\frac{J!}{(J-(L-n))!}$ can be replaced by J^{L-n}, up to corrections of order $o(\frac{L}{J})$. Therefore

$$\begin{aligned} \left\{ \begin{array}{ccc} \alpha & \alpha & L \\ \alpha & \alpha & J \end{array} \right\} &\approx \frac{(-1)^{2\alpha+J}}{2\alpha} \left(\frac{J}{2\alpha} \right)^{2L} \sum_{n=0}^{L} (-1)^n \left(\begin{array}{c} L \\ n \end{array} \right)^2 \left((\frac{2\alpha}{J})^2 - 1 \right)^n \\ &= \frac{(-1)^{2\alpha+J}}{2\alpha} P_L(1 - \frac{J^2}{2\alpha^2}), \end{aligned} \tag{4-51}$$

   as claimed.

2. $J \ll \alpha$:

   Then in the sum (4-50), the dominant term is $n = L$, because $\frac{J!}{(J-(L-n))!} \leq J^{L-n}$. Therefore one can safely replace the term $\frac{J!}{(J-(L-n))!}$ in this sum by its value at $n = L$, namely J^{L-n}. The remaining terms are smaller by a factor of $(\frac{J}{\alpha})^2$. Hence we can continue as in case 1.

If $J \leq L$, one can either use the same argument as in the 2nd case since the term $n = L$ is dominant, or use the symmetry of the $6j$ symbols in L, J together with $P_L(1-\frac{J^2}{2\alpha^2}) \approx P_J(1-\frac{L^2}{2\alpha^2})$ for $J, L \ll \alpha$. Finally if $J + L \geq 2\alpha$, then the term $n = 0$ dominates, and one can proceed as in case 1. Therefore (4-31) is valid for all $0 \leq J \leq 2\alpha$.

One can illustrate the excellent approximation for the $6j$ symbols provided by (4-31) for all $0 \leq J \leq 2\alpha$ using numerical calculations.

Identities for $3j$ and $6j$ symbols. We quote here some identities of the $3j$ and $6j$ symbols which are used to derive the expressions (4-22) and (4-23) for the one–loop corrections. The $3j$ symbols satisfy the orthogonality relation

$$\sum_{j,l} \begin{pmatrix} J & L & K \\ j & l & k \end{pmatrix} \begin{pmatrix} J & L & K' \\ -j & -l & -k' \end{pmatrix} = \frac{(-1)^{K-L-J}}{2K+1} \delta_{K,K'} \delta_{k,k'}, \tag{4-52}$$

assuming that (J, L, K) form a triangle.

The $6j$ symbols satisfy standard symmetry properties, and the orthogonality relation

$$\sum_N (2N+1) \begin{Bmatrix} A & B & N \\ C & D & P \end{Bmatrix} \begin{Bmatrix} A & B & N \\ C & D & Q \end{Bmatrix} = \frac{1}{2P+1} \delta_{P,Q}, \tag{4-53}$$

assuming that (A, D, P) and (B, C, P) form a triangle. Furthermore, the following sum rule is used in (4-23)

$$\sum_N (-1)^{N+P+Q} (2N+1) \begin{Bmatrix} A & B & N \\ C & D & P \end{Bmatrix} \begin{Bmatrix} A & B & N \\ D & C & Q \end{Bmatrix} = \begin{Bmatrix} A & C & Q \\ B & D & P \end{Bmatrix}. \tag{4-54}$$

The Biedenharn–Elliott relations are needed to verify associativity of (4-6):

$$\sum_N (-1)^{N+S} (2N+1) \begin{Bmatrix} A & B & N \\ C & D & P \end{Bmatrix} \begin{Bmatrix} C & D & N \\ E & F & Q \end{Bmatrix} \begin{Bmatrix} E & F & N \\ B & A & R \end{Bmatrix} = \begin{Bmatrix} P & Q & R \\ E & A & D \end{Bmatrix} \begin{Bmatrix} P & Q & R \\ F & B & C \end{Bmatrix}, \tag{4-55}$$

where $S = A + B + C + D + E + F + P + Q + R$. All these can be found e.g. in [108].

Chapter 5

Second quantization on the q-deformed fuzzy sphere

We have seen in the example of the fuzzy sphere that field theory can be q-deformed, in a more or less straightforward manner. Other q–deformed field theories have been considered before, see for example [85, 114, 115, 116, 117, 118, 119] and references therein, mainly for scalar fields though. However, they should be considered as "classical" field theory on some kind of nonlocal space, rather than a quantized field theory in the physical sense, where each mode of a field should be an operator on a Hilbert space or equivalently all possible configurations should be integrated over via a Feynman path integral. In this final chapter, we proceed to (euclidean) *quantum* field theory on the q–deformed fuzzy sphere $\mathcal{S}^2_{q,N}$. It is based on a paper [38] written in collabration with Harald Grosse and John Madore.

The second quantization of q-deformed field theories has proved to be difficult. The main problem is probably the apparent incompatibility between the symmetrization postulate of quantum field theory (QFT) which involves the permutation group, and the fact that quantum groups are naturally related to the braid group rather than the permutation group. One could of course consider theories with generalized statistics; however if q–deformation is considered as a "deformation" of ordinary geometry, then it should be possible to define models with a smooth limit $q \to 1$. In particular, the degrees of freedom should be independent of q. This is the guiding principle of the present approach, together with covariance under the quantum group of motions $U_q(su(2))$: our goal is to define a q–deformed (euclidean) quantum field theory which is essentially bosonic, and has a smooth limit $q \to 1$ as an ordinary quantum field theory. Moreover, we would like to have a map from the q-deformed quantum field theory to some undeformed, but nonlocal theory, i.e. some kind of Seiberg-Witten map. This is also expected from the point of view of string theory [5]. While some proposals have been given in the literature [120, 121] how to define quantum field theories on spaces with quantum group symmetry, none of them seems to satisfies these requirements.

We will show how to accomplish this goal using a path integral approach, integrating over all modes or harmonics of the field. To this end, it turns out to be useful to define a quasiassociative

star–product of the modes, based on the Drinfeld twist. Quasiassociativity appears only in intermediate steps of the mathematical formalism, and is not at all in contradiction with the axioms of quantum mechanics. Indeed we provide also an equivalent formulation which is entirely within the framework of associative algebras, and in particular we give in Section 5.6.4 an operator formulation of scalar field theory in $2_q + 1$ dimensions. The latter is very instructive to understand how e.g. bosons and quantum groups can coexist.

The models we find have a manifest $U_q(su(2))$ symmetry with a smooth limit $q \to 1$, and satisfy positivity and twisted bosonic symmetry properties. We also develop some of the standard tools of quantum field theory, in particular we give a systematic way to calculate n–point correlators in perturbation theory. As applications of the formalism, the 4–point correlator of a free scalar field theory is calculated, as well as the planar contribution to the tadpole diagram in a ϕ^4 theory. Gauge fields are discussed in Section 5.6.3. Here the correct quantization is less clear at present, and we only suggest 2 possible variants of a path integral quantization.

We should point out that while only the q-deformed fuzzy sphere $\mathcal{S}^2_{q,N}$ is considered here, the proposed quantization prodecure is not restricted to this case, and not even to 2 dimensions. This chapter is the result of a long period of trial-and-error trying to q-deform quantum field theory with the above requirements. $\mathcal{S}^2_{q,N}$ is particularly well suited to attack the problem of second quantization, because there is only a finite number of modes. This means that all considerations can be done on a purely algebraic level, and are essentially rigorous. However, our constructions can be applied in principle to any other q-deformed space, provided the decomposition of the fields in terms of irreducible representations of the underlying quantum group is known. Furthermore, it requires the knowledge of some rather involved group-theoretical objects (such as coassociators) build from Drinfeld twists. While the latter are not needed explicitly, much work is still needed before for example a full loop calculation becomes possible.

5.1 Why QFT on q-deformed spaces is difficult

To understand the problem, consider scalar fields, which are elements $\psi \in \mathcal{S}^2_{q,N}$. A reasonable action could have the form

$$S[\psi] = -\int\limits_{\mathcal{S}^2_{q,N}} \psi\Delta\psi + \lambda\psi^4, \tag{5-1}$$

where Δ is the Laplacian [36]. Such actions are invariant under the quantum group $U_q(su(2))$ of rotations, and they are real, $S[\psi]^* = S[\psi]$. They define a first–quantized euclidean scalar field theory on the q–deformed fuzzy sphere.

We want to study the second quantization of these models. On the undeformed fuzzy sphere, this is fairly straightforward [61, 29]: The fields can be expanded in terms of irreducible representations of $SO(3)$,

$$\psi(x) = \sum_{K,n} \psi_{K,n}(x)\; a^{K,n} \tag{5-2}$$

with coefficients $a^{K,n} \in \mathbb{C}$. The above actions then become polynomials in the variables $a^{K,n}$ which are invariant under $SO(3)$, and the "path integral" is naturally defined as the product of the ordinary integrals over the coefficients $a^{K,n}$. This defines a quantum field theory which has a $SO(3)$ rotation symmetry, because the path integral is invariant.

In the q–deformed case, this is not so easy. The reason is that the coefficients $a^{K,n}$ in (5-2) must be considered as representations of $U_q(su(2))$ in order to have such a symmetry at the quantum level. This implies that they cannot be ordinary complex numbers, because a commutative algebra is not consistent with the action of $U_q(su(2))$, whose coproduct is not cocommutative. Therefore an ordinary integral over commutative modes $a^{K,n}$ would violate $U_q(su(2))$ invariance at the quantum level. On the other hand, no associative algebra with generators $a^{K,n}$ is known (except for some simple representations) which is both covariant under $U_q(su(2))$ and has the same Poincaré series as classically, i.e. the dimension of the space of polynomials at a given degree is the same as in the undeformed case. The latter is an essential physical requirement at least for low energies, in order to have the correct number of degrees of freedom, and is usually encoded in a symmetrization postulate. It means that the "amount of information" contained in the n–point functions should be the same as for $q = 1$. These issues will be discussed on a more formal level in Section 5.5. While some proposals have been given in the literature [120, 121] how to define QFT on spaces with quantum group symmetry, none of them seems to satisfies all these requirements.

One possible way out was suggested in [122], where it was pointed out that a symmetrization can be achieved using a Drinfeld twist, at least in any given n–particle sector. Roughly speaking, the Drinfeld twist relates the tensor product of representations of quantum groups to the tensor product of undeformed ones, and hence essentially allows to use the usual completely symmetric Hilbert space. The problem remained, however, how to treat sectors with different particle number simultaneously, which is essential for a QFT, and how to handle the Drinfeld twists which are very difficult to calculate.

We present here a formalism which solves these problems, by defining a star product of the modes $a^{K,n}$ which is covariant under the quantum group, and in the limit $q \to 1$ reduces to the commutative algebra of functions in the $a^{K,n}$. This algebra is quasiassociative, but satisfies all the requirements discussed above. In particular, the number of independent polynomials in the $a^{K,n}$ is the same as usual. One can then define an invariant path integral, which yields a consistent and physically reasonable definition of a second–quantized field theory with a quantum group symmetry. In particular, the "correlation functions" will satisfy invariance, hermiticity, positivity and symmetry properties. An essentially equivalent formulation in terms of a slightly extended associative algebra will be presented as well, based on constructions by Fiore [123]. It turns out to be related to the general considerations in [80]. The appearance of quasiassociative algebras is also reminiscent of results in the context of D–branes on WZW models [8, 124].

Our considerations are not restricted to 2 dimensions, and should be applicable to other spaces with quantum group symmetry as well. The necessary mathematical tools will be developed in Sections 5.2, 5.3, and 5.4. After discussing the definition and basic properties of QFT on $\mathcal{S}^2_{q,N}$

in Section 5.5, we derive formulas to calculate n–point functions in perturbation theory, and find an analog of Wick's theorem. All diagrams on $\mathcal{S}^2_{q,N}$ are of course finite, and vacuum diagrams turn out to cancel a usual. The resulting models can also be interpreted as field theories on the undeformed fuzzy sphere, with slightly "nonlocal" interactions.

As applications of the general method, we consider first the case of a free scalar field theory, and calculate the 4–point functions. The tadpole diagram for a ϕ^4 model is studied as well, and turns out to be linearly divergent as $N \to \infty$. We then discuss two possible quantizations of gauge models, and finally consider scalar field theory on $\mathcal{S}^2_{q,N}$ with an extra time.

We should stress that our approach is quite conservative, as it aims to find a "deformation" of standard quantum field theory in a rather strict sense, with ordinary statistics. Of course on can imagine other, less conventional approaches, such as the one in [120]. Moreover, we only consider the case $q \in \mathbb{R}$ in this paper. It should be possible to modify our methods so that the case of q being a root of unity can also be covered. Then QFT on more realistic spaces such as 4–dimensional quantum Anti–de Sitter space [78] could be considered as well. There, the number of modes as well as the dimensions of the relevant representations are finite at roots of unity, as in the present paper.

5.2 More on Drinfeld twists

We first have to extend our knowledge of Drinfeld twists, which were briefly introduced in Chapter 1. In order to avoid confusions, the language will be quite formal initially. To a given a finite–dimensional simple Lie algebra g (for our purpose just $su(2)$), one can associate 2 Hopf algebras [24, 43, 42]: the usual $(U(g)[[h]], m, \varepsilon, \Delta, S)$, and the q–deformed $(U_q(g), m_q, \varepsilon_q, \Delta_q, S_q)$. Here $U(g)$ is the universal enveloping algebra, $U_q(g)$ is the q–deformed universal enveloping algebra, and $U(g)[[h]]$ are the formal power series in h with coefficients in $U(g)$. The symbol

$$q = e^h$$

is considered formal for now. As already discussed in Section 1.4.5, a theorem by Drinfeld states that there exists an algebra isomorphism

$$\varphi : U_q(g) \to U(g)[[h]] \tag{5-3}$$

and a 'twist', i.e. an element

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \ \in\ U(g)[[h]] \otimes U(g)[[h]]$$

(in a Sweedler notation, where a sum is implicitly understood) satisfying

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)\mathcal{F}, \tag{5-4}$$

$$\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + o(h), \tag{5-5}$$

which relates these two Hopf algebra $U_q(g)$ and $U(g)[[h]]$ as follows: if $\mathcal{F}^{-1} = \mathcal{F}_1^{-1} \otimes \mathcal{F}_2^{-1}$ is the inverse[1] of $\mathcal{F}$, then

$$\begin{aligned} \varphi(m_q) &= m \circ (\varphi \otimes \varphi), & \text{(5-6)} \\ \varepsilon_q &= \varepsilon \circ \varphi, & \text{(5-7)} \\ \varphi(S_q(u)) &= \gamma^{-1} S(\varphi(u))\gamma, & \text{(5-8)} \\ \varphi(S_q^{-1}(u)) &= \gamma' S(\varphi(u))\gamma'^{-1}, & \text{(5-9)} \\ (\varphi \otimes \varphi)\Delta_q(u) &= \mathcal{F}\Delta(\varphi(u))\mathcal{F}^{-1}, & \text{(5-10)} \\ (\varphi \otimes \varphi)\mathcal{R} &= \mathcal{F}_{21} q^{\frac{t}{2}} \mathcal{F}^{-1}. & \text{(5-11)} \end{aligned}$$

for any $u \in U_q(g)$. Here $t := \Delta(C) - \mathbf{1} \otimes C - C \otimes \mathbf{1}$ is the canonical invariant element in $U(g) \otimes U(g)$, C is the quadratic Casimir, and

$$\gamma = S(\mathcal{F}_1^{-1})\mathcal{F}_2^{-1}, \qquad \gamma' = \mathcal{F}_2 S \mathcal{F}_1 \tag{5-12}$$

Moreover, $\gamma^{-1}\gamma'$ is central in $U(g)[[h]]$. The undeformed maps[2] $m, \varepsilon, \Delta, S$ have been linearly extended from $U(g)$ to $U(g)[[h]]$; notice that $S^2 = 1$. $\mathcal{F}_{21}$ is obtained from $\mathcal{F}$ by flipping the tensor product. This kind of notation will be used throughout from now on. Coassociativity of Δ_q follows from the fact that the (nontrivial) coassociator

$$\phi := [(\Delta \otimes \mathrm{id})\mathcal{F}^{-1}](\mathcal{F}^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)\mathcal{F}] \tag{5-13}$$

is $U(g)$–invariant, i.e.

$$[\phi, \Delta^{(2)}(u)] = 0$$

for $u \in U(g)$. Here $\Delta^{(2)}$ denotes the usual 2–fold coproduct.

In the present paper, we only consider finite–dimensional representations, i.e. operator algebras rather than the abstract ones. Then the formal parameter $q = e^h$ can be replaced by a real number close to 1, and all statements in this section still hold since the power series will converge. One could then identify the algebras $U(su(2))$ with $U_q(su(2))$ (but not as coalgebras!) via the the isomorphism φ. We will usually keep φ explicit, however, in order to avoid confusions.

It turns out that the twist $\mathcal{F}$ is not determined uniquely, but there is some residual "gauge freedom" [56, 55],

$$\mathcal{F} \to \mathcal{F}T \tag{5-14}$$

with an arbitrary symmetric $T \in U(g)[[h]]^{\otimes 2}$ which commutes with $\Delta(U(g))$ and satisfies (5-4), (5-5). The symmetry of T guarantees that $\mathcal{R}$ is unchanged, so that $\mathcal{F}$ remains a twist from $(U(g)[[h]], m, \varepsilon, \Delta, S)$ to $(U_q(q), m_q, \varepsilon_q, \Delta_q, S_q)$. We will take advantage of this below.

While for the twist $\mathcal{F}$, little is known apart apart from its existence, one can show [123] using results of Kohno [125, 126] and Drinfeld [56, 55] that the twists can be chosen such that the

[1] it exists as a formal power series because of (5-5)

[2] we will suppress the multiplication maps from now on

following formula holds:

$$\phi = \lim_{x_0,y_0 \to 0^+} \left\{ x_0^{-\frac{h}{2\pi i} t_{12}} \vec{P}\exp\left[-\frac{h}{2\pi i} \int\limits_{x_0}^{1-y_0} dx \left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1} \right) \right] y_0^{\frac{h}{2\pi i} t_{23}} \right\} = \mathbf{1} + o(h^2). \quad \text{(5-15)}$$

Here $\vec{P}$ denotes the path–ordered exponential. Such twists were called "minimal" by Fiore [123], who showed that they satisfy the following remarkable relations:

$$\begin{aligned} \mathbf{1} &= \mathcal{F}\Delta\mathcal{F}_1(1 \otimes (S\mathcal{F}_2)\gamma), && \text{(5-16)} \\ &= (1 \otimes S\mathcal{F}_2{\gamma'}^{-1})\mathcal{F}(\Delta\mathcal{F}_1) && \text{(5-17)} \\ &= \mathcal{F}\Delta\mathcal{F}_2((S\mathcal{F}_1){\gamma'}^{-1} \otimes 1) && \text{(5-18)} \\ &= (\gamma^{-1} S\mathcal{F}_1^{-1} \otimes 1)\Delta\mathcal{F}_2^{-1}\mathcal{F}^{-1} && \text{(5-19)} \\ &= \Delta\mathcal{F}_2^{-1}\mathcal{F}^{-1}(\gamma' S\mathcal{F}_1^{-1} \otimes 1) && \text{(5-20)} \\ &= (1 \otimes \gamma' S\mathcal{F}_2^{-1})\Delta\mathcal{F}_1^{-1}\mathcal{F}^{-1} && \text{(5-21)} \end{aligned}$$

All coproducts here are undeformed. For such twists, one can write down inverses of the elements γ, γ':

$$\gamma^{-1} = \mathcal{F}_1 S\mathcal{F}_2 = S\gamma', \qquad {\gamma'}^{-1} = S(\mathcal{F}_2^{-1})\mathcal{F}_1^{-1} = S\gamma. \quad \text{(5-22)}$$

Furthermore, we add the following observation: let $(V_i, \triangleright)$ be representations of $U(g)$ and $I^{(3)} \in V_1 \otimes V_2 \otimes V_3$ be an invariant tensor, so that $u \triangleright I^{(3)} \equiv \Delta^{(2)}(u) \triangleright I^{(3)} = \varepsilon(u) I^{(3)}$ for $u \in U(g)$. Then the (component–wise) action of ϕ on $I^{(3)}$ is trivial:

$$\phi \triangleright I^{(3)} = I^{(3)}. \quad \text{(5-23)}$$

This follows from (5-15): observe that t_{12} commutes with t_{23} in the exponent, because e.g. $(\Delta(C) \otimes \mathbf{1})$ can be replaced by $\mathbf{1} \otimes \mathbf{1} \otimes C$ if acting on invariant tensors. Therefore the path–ordering becomes trivial, and (5-23) follows.

Star structure. Consider on $U(su(2))[[h]]$ the (antilinear) star structure

$$H^* = H, \quad X^{\pm *} = X^{\mp}, \quad \text{(5-24)}$$

with $h^* = h$, since q is real. It follows e.g. from its explicit form [127, 128] that the algebra map φ is compatible with this star,

$$\varphi(u)^* = \varphi(u^*).$$

It was shown in [129] that using a suitable gauge transformation (5-14), it is possible to choose $\mathcal{F}$ such that it is unitary,

$$(* \otimes *)\mathcal{F} = \mathcal{F}^{-1}. \quad \text{(5-25)}$$

Moreover, it was stated in [123] without proof that the following stronger statement holds:

Proposition 5.2.1. *Using a suitable gauge transformation (5-14), it is possible to choose a twist $\mathcal{F}$ which for $q \in \mathbb{R}$ is both unitary and minimal, so that (5-25) and (5-16) to (5-21) hold.*

Since this is essential for us, we provide a proof in Section 5.7.

5.3 Twisted $U_q(g)$–covariant $\star$ –product algebras

Let $(\mathcal{A}, \cdot, \triangleright)$ be an associative $U(g)$–module algebra, which means that there exists an action

$$\begin{aligned} U(g) \times \mathcal{A} &\rightarrow \mathcal{A}, \\ (u, a) &\mapsto u \triangleright a \end{aligned}$$

which satisfies $u \triangleright (ab) = (u_{(1)} \triangleright a)(u_{(2)} \triangleright b)$ for $a, b \in \mathcal{A}$. Here $\Delta(u) = u_{(1)} \otimes u_{(2)}$ denotes the undeformed coproduct. Using the map φ (5-13), we can then define an action of $U_q(g)$ on $\mathcal{A}$ by

$$u \triangleright_q a := \varphi(u) \triangleright a, \tag{5-26}$$

or $u \triangleright_q a_i = a_j \pi^j_i(\varphi(u))$ in matrix notation. This does *not* define a $U_q(g)$–module algebra, because the multiplication is not compatible with the coproduct of $U_q(g)$. However, one can define a new multiplication on $\mathcal{A}$ as follows (1-50):

$$\boxed{a \star b := (\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b) = \cdot(\mathcal{F}^{-1} \triangleright (a \otimes b))} \tag{5-27}$$

for any $a, b \in \mathcal{A}$. It is well–known [41] that $(\mathcal{A}, \star, \triangleright_q)$ is now a $U_q(g)$–module algebra:

$$\begin{aligned} u \triangleright_q (a \star b) &= \varphi(u) \triangleright \big((\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b)\big) \\ &= \cdot\big((\Delta(\varphi(u))\mathcal{F}^{-1}) \triangleright a \otimes b\big) \\ &= \cdot\big((\mathcal{F}^{-1}(\varphi \otimes \varphi)\Delta_q(u)) \triangleright a \otimes b\big) \\ &= \star\,(\Delta_q(u) \triangleright_q a \otimes b) \end{aligned}$$

for $u \in U_q(g)$. In general, this product $\star$ is not associative, but it is *quasiassociative*, which means that

$$(a \star b) \star c = (\tilde{\phi}_1 \triangleright a) \star \Big((\tilde{\phi}_2 \triangleright b) \star (\tilde{\phi}_3 \triangleright c)\Big)\,. \tag{5-28}$$

where

$$\tilde{\phi} := (\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)\mathcal{F}][(\Delta \otimes \mathrm{id})\mathcal{F}^{-1}](\mathcal{F}^{-1} \otimes \mathbf{1}) = \mathrm{U}_{\mathcal{F}}\; \phi\; \mathrm{U}_{\mathcal{F}}^{-1} \tag{5-29}$$

with

$$U_{\mathcal{F}} = (\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)\mathcal{F}] \quad \in \mathrm{U(g)}^{\otimes 3},$$

which satisfies

$$[\tilde{\phi}, \Delta_q^{(2)}(u)] = 0$$

for $u \in U_q(g)$. All this follows immediately from the definitions. Moreover, the following simple observation will be very useful:

Lemma 5.3.1. *In the above situation,*

$$(a \star b) \star c = a \star (b \star c) \tag{5-30}$$

if one of the factors $a, b, c \in \mathcal{A}$ is invariant under $U(g)$. If $(\mathcal{A}, \cdot)$ is commutative, then any element $S \in \mathcal{A}$ which is invariant under the action of $U(g)$, $u \triangleright S = \varepsilon(u)\, S$, is central in $(\mathcal{A}, \star, \triangleright_q)$

Note that invariance of an element $a \in \mathcal{A}$ under $U(g)$ is the same as invariance under $U_q(g)$.

Proof. This follows immediately from (5-4) together with the definition of $\tilde{\phi}$. To see the last statement, assume that S is invariant. Then

$$\begin{aligned} S \star a &= (\mathcal{F}_1^{-1} \triangleright S) \cdot (\mathcal{F}_2^{-1} \triangleright a) \\ &= \cdot \big(((\varepsilon \otimes \mathbf{1})\mathcal{F}^{-1}) \triangleright (S \otimes a)\big) \\ &= S \cdot a = a \cdot S \\ &= a \star S \end{aligned} \tag{5-31}$$

for any $a \in \mathcal{A}$. □

For actual computations, it is convenient to use a tensor notation as follows: assume that the elements $\{a_i\}$ of $\mathcal{A}$ form a representation of $U(g)$. Denoting $\tilde{\phi}^{rst}_{ijk} = \pi^r_i(\tilde{\phi}_1)\, \pi^s_j(\tilde{\phi}_2)\pi^t_k(\tilde{\phi}_3)$, equation (5-28) can be written as

$$\begin{aligned} (a_i \star a_j) \star a_k &= a_r \star (a_s \star a_t)\tilde{\phi}^{rst}_{ijk}, \quad \text{or} \\ (a_1 \star a_2) \star a_3 &= a_1 \star (a_2 \star a_3)\, \tilde{\phi}_{123}. \end{aligned} \tag{5-32}$$

The last notation will always imply a matrix multiplication as above.

Conversely, given a $U_q(g)$–module algebra $(\mathcal{A}, \star, \triangleright_q)$, one can twist it into a $U(g)$–module algebra $(\mathcal{A}, \cdot, \triangleright)$ by

$$a \cdot b := (\varphi^{-1}(\mathcal{F}^{(1)}) \triangleright_q a) \star (\varphi^{-1}(\mathcal{F}^{(2)}) \triangleright_q b)$$

where of course $u \triangleright a = \varphi^{-1}(u) \triangleright_q a$. Now if $(\mathcal{A}, \star, \triangleright_q)$ was associative, then $(\mathcal{A}, \cdot, \triangleright)$ is quasiassociative,

$$a \cdot (b \cdot c) = \phi \triangleright_q^{(3)} ((a \cdot b) \cdot c) := ((\phi_1 \triangleright_q a) \cdot (\phi_2 \triangleright_q b)) \cdot (\phi_3 \triangleright_q c).$$

Such a twist was used in [36] to obtain the associative algebra of functions on the q–deformed fuzzy sphere from the quasi–associative algebra of functions on $D2$–branes in the $SU(2)$ WZW model found in [8].

Commutation relations and $\mathcal{R}$–matrices. These twisted algebras have a more intrinsic characterization, which is much more practical. Consider a commutative $U(g)$–module algebra $(\mathcal{A}, \cdot, \triangleright)$, and the associated twisted $U_q(g)$–module algebra $(\mathcal{A}, \star, \triangleright_q)$ as defined above. Observe that the definition (5-27) is equivalent to

$$\begin{aligned} a \star b &= (\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b) = (\mathcal{F}_2^{-1} \triangleright b) \cdot (\mathcal{F}_1^{-1} \triangleright a) \\ &= \cdot \big((\mathcal{F}^{-1}\mathcal{F}\mathcal{F}_{21}^{-1}) \triangleright (b \otimes a)\big) \\ &= (\tilde{\mathcal{R}}_2 \triangleright_q b) \star (\tilde{\mathcal{R}}_1 \triangleright_q a) \end{aligned} \tag{5-33}$$

where we define

$$\tilde{\mathcal{R}} := (\varphi^{-1} \otimes \varphi^{-1})\mathcal{F}_{21}\mathcal{F}^{-1} = \tilde{\mathcal{R}}_{21}^{-1}. \tag{5-34}$$

In a given representation, this can be written as

$$a_i \star a_j = a_k \star a_l \, \tilde{\mathcal{R}}^{lk}_{ij}, \quad \text{or} \quad a_1 \star a_2 = a_2 \star a_1 \, \tilde{\mathcal{R}}_{12} \tag{5-35}$$

where

$$\tilde{\mathcal{R}}^{ji}_{kl} = (\pi^j_k \otimes \pi^i_l)(\tilde{\mathcal{R}}). \tag{5-36}$$

Now there is no more reference to the "original" $U(g)$–covariant algebra structure. $\tilde{\mathcal{R}}$ does not satisfy the quantum Yang–Baxter equation in general, which reflects the non–associativity of the $\star$ product. However it does satisfy

$$\begin{aligned}
\tilde{\mathcal{R}}\tilde{\mathcal{R}}_{21} &= \mathbf{1}, && (5\text{-}37)\\
\tilde{\mathcal{R}}_{(12),3} := (\Delta_q \otimes 1)\tilde{\mathcal{R}} &= \tilde{\phi}_{312}\tilde{\mathcal{R}}_{13}\tilde{\phi}^{-1}_{132}\tilde{\mathcal{R}}_{23}\tilde{\phi}_{123} && (5\text{-}38)\\
\tilde{\mathcal{R}}_{1,(23)} := (1 \otimes \Delta_q)\tilde{\mathcal{R}} &= \tilde{\phi}^{-1}_{231}\tilde{\mathcal{R}}_{13}\tilde{\phi}_{213}\tilde{\mathcal{R}}_{12}\tilde{\phi}^{-1}_{123}, && (5\text{-}39)
\end{aligned}$$

as can be verified easily. This means that we are working with the quasitriangular quasi–Hopf algebra [56, 55] $(U_q(g), \Delta_q, \tilde{\phi}, \tilde{\mathcal{R}})$, which is obtained from the ordinary Hopf algebra $(U(g), \Delta, \mathbf{1}, \mathbf{1})$ by the Drinfeld twist $\mathcal{F}$. In practice, it is much easier to work with $\tilde{\mathcal{R}}$ than with $\mathcal{F}$. For $q \in \mathbb{R}$, one can in fact write

$$\tilde{\mathcal{R}} = \mathcal{R}\, \sqrt{\mathcal{R}_{21}\mathcal{R}_{12}}^{\,-1}, \tag{5-40}$$

where $\mathcal{R}$ is the usual universal R –matrix (5-11) of $U_q(g)$, which does satisfy the quantum Yang–Baxter equation. The product $(\mathcal{R}_{21}\mathcal{R}_{12})$ could moreover be expressed in terms of the Drinfeld–Casimir

$$v = (S_q\mathcal{R}_2)\mathcal{R}_1 q^{-H}, \tag{5-41}$$

which is central in $U_q(g)$ and satisfies $\Delta(v) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1} v \otimes v$. The square root is well–defined on all the representations which we consider, since q is real.

Twisted Heisenberg algebras. Consider the $U(g)$–module algebra $(\mathcal{A}_H, \cdot, \triangleright)$ with generators a_i and $a^\dagger_j$ in some given irreducible representation and commutation relations

$$\begin{aligned}
[a^\dagger_i, a^\dagger_j] &= 0 = [a_i, a_j],\\
\left[a^\dagger_i, a_j\right] &= (g_c)_{ij}
\end{aligned} \tag{5-42}$$

where $(g_c)_{ij}$ is the (unique) invariant tensor in the given representation of $U(g)$. We can twist $(\mathcal{A}_H, \cdot, \triangleright)$ as above, and obtain the $U_q(g)$–module algebra $(\mathcal{A}_H, \star, \triangleright_q)$. The new commutation relations among the generators can be evaluated easily:

$$\begin{aligned}
a_1 \star a_2 &= a_2 \star a_1 \, \tilde{\mathcal{R}}_{12},\\
a^\dagger_1 \star a^\dagger_2 &= a^\dagger_2 \star a^\dagger_1 \, \tilde{\mathcal{R}}_{12},\\
a^\dagger_1 \star a_2 &= g_{12} + \, a_2 \star a^\dagger_1 \, \tilde{\mathcal{R}}_{12}.
\end{aligned} \tag{5-43}$$

Here

$$g_{nm} = (g_c)_{rs}\ \pi^r_n(\mathcal{F}_1^{-1})\pi^s_m(\mathcal{F}_2^{-1}) \tag{5-44}$$

is the unique rank 2 tensor which is invariant under the action $\triangleright_q$ of $U_q(g)$. A similar relation holds for the invariant tensor with upper indices:

$$g^{nm} = \pi^n_r(\mathcal{F}_1)\pi^m_s(\mathcal{F}_2)\ g_c^{rs}, \tag{5-45}$$

which satisfies $g^{nm}g_{ml} = \delta^n_l$. In particular, it follows that

$$a \star a := a_1 \star a_2\ g^{12} = a_1 \cdot a_2\ (g_c)^{12}, \tag{5-46}$$

therefore the invariant bilinears remain undeformed. This is independent of the algebra of the generators a_i.

It is sometimes convenient to use the q–deformed antisymmetrizer [122]

$$P^-_{12} = \mathcal{F}_{12}(1-\delta^{21}_{12})\mathcal{F}^{-1}_{12} = (\mathbf{1} - P\tilde{\mathcal{R}})_{12}$$

acting on the tensor product of 2 identical representations, where P is the flip operator. Then the commutation relations (5-35) can be written as

$$a_1 \star a_2\ P^-_{12} = 0. \tag{5-47}$$

For products of 3 generators, the following relations hold:

Lemma 5.3.2.

$$\begin{aligned} a_1 \star (a_2 \star a_3) &= (a_2 \star a_3) \star a_1\ \tilde{\mathcal{R}}_{1,(23)}, & (5\text{-}48)\\ a_1^\dagger \star (a_2 \star a_3\ g^{23}) &= 2a_1 + (a_2 \star a_3\ g^{23}) \star a_1^\dagger, & (5\text{-}49)\\ a_1^\dagger \star (a_2 \star a_3\ P^-_{23}) &= (a_2 \star a_3\ P^-_{23}) \star a_1^\dagger\ \tilde{\mathcal{R}}_{1,(23)}. & (5\text{-}50) \end{aligned}$$

The proof is in Section 5.7.

5.3.1 Integration

From now on we specialize to $g = su(2)$, even though much of the following holds more generally. Let $\{a_i\}$ be a basis of the spin K representation of $U(su(2))$ with integer K, and consider the (free) commutative algebra $\mathcal{A}$ generated by these variables. Let g_c^{ij} be the (real, symmetric) invariant tensor, so that $a \cdot a := a_i a_j g_c^{ij}$ is invariant under $U(su(2))$, and $g_c^{ij} g_c^{jk} = \delta^{ik}$. We now impose on $\mathcal{A}$ the star structure

$$a_i^* = g_c^{ij} a_j, \tag{5-51}$$

so that $\mathcal{A}$ can be interpreted as the algebra of complex–valued functions on $\mathbb{R}^{2K+1}$; in particular, $a \cdot a$ is real. Then the usual integral on $\mathbb{R}^{2K+1}$ defines a functional on (the subset of integrable functions in a suitable completion of) $\mathcal{A}$, which satisfies

$$\begin{aligned} \int d^{2K+1}a \; u \triangleright f &= \varepsilon(u) \int d^{2K+1}a \; f, \\ \left(\int d^{2K+1}a \; f\right)^* &= \int d^{2K+1}a \; f^* \end{aligned} \tag{5-52}$$

for $u \in U(su(2))$ and integrable $f \in \mathcal{A}$. More general invariant functionals on $\mathcal{A}$ can be defined as

$$\langle f \rangle := \int d^{2K+1}a \; \rho(a \cdot a) \; f \tag{5-53}$$

for $f \in \mathcal{A}$, where ρ is a suitable real weight function. They are invariant, real and positive:

$$\begin{aligned} \langle u \triangleright f \rangle &= \varepsilon(u)\langle f \rangle, \\ \langle f \rangle^* &= \langle f^* \rangle \\ \langle f^* f \rangle &\geq 0 \end{aligned} \tag{5-54}$$

for any $u \in U(su(2))$ and $f \in \mathcal{A}$. As usual, one can then define a Hilbert space of square–(weight–) integrable functions by

$$\langle f, g \rangle := \langle f^* g \rangle = \int d^{2K+1}a \; \rho(a \cdot a) \; f^* g. \tag{5-55}$$

Now consider the twisted $U_q(su(2))$–module algebra $(\mathcal{A}, \star, \triangleright_q)$ defined in the previous section. We want to find an integral on $\mathcal{A}$ which is invariant under the action $\triangleright_q$ of $U_q(su(2))$. Formally, this is very easy: since the *space* $\mathcal{A}$ is unchanged by the twisting, we can simply use the classical integral again, and verify invariance

$$\int d^{2K+1}a \; u \triangleright_q f = \int d^{2K+1}a \; \varphi(u) \triangleright f = \varepsilon(\varphi(u)) \int d^{2K+1}a \; f = \varepsilon_q(u) \int d^{2K+1}a \; f.$$

Notice that the algebra structure of $\mathcal{A}$ does not enter here at all. The compatibility with the reality structure will be discussed in the next section.

Of course we have to restrict to certain classes of integrable functions. However, this is not too hard in the cases of interest. Consider for example the space of Gaussian functions, i.e. functions of the form $P(a_i)e^{-c(a \cdot a)}$ with suitable (polynomial, say) $P(a_i)$. Using (5-46), this is the same as the space of Gaussian functions in the sense of the star product, $P_\star(a_i)e^{-c(a \star a)}$. This will imply that all integrals occuring in perturbation theory are well–defined. Furthermore, one can obtain a twisted sphere by imposing the relation $a \star a = a \cdot a = R^2$. On this sphere, the integral is well–defined for any polynomial functions. The integral over the twisted $\mathbb{R}^{2K+1}$ can hence be calculated by first integrating over the sphere and then over the radius. Finally, we point out the following obvious fact:

$$\langle P(a) \rangle = \langle P_0(a) \rangle \tag{5-56}$$

where $P_0(a) \in \mathcal{A}$ is the singlet part of the decomposition of the polynomial $P(a)$ under the action $\triangleright_q$ of $U_q(su(2))$, or equivalently under the action $\triangleright$ of $U(su(2))$.

5.4 An $U_q(su(2))$–covariant operator formalism

In the previous section, we defined quasi–associative algebras of functions on arbitrary representation spaces of $U_q(su(2))$. We will apply this to the coefficients of the fields on $\mathcal{S}^2_{q,N}$ later. However, there is an alternative approach within the framework of ordinary operators and representations, which is essentially equivalent for our purpose. We shall follow here closely the constructions in [123]. It seems that both approaches have their own advantages, therefore we want to discuss them both.

We first recall the notion of the semidirect product (cross–product) algebra, which is useful here. Let $(\mathcal{A}, \cdot, \triangleright)$ be an associative $U(su(2))$–module algebra. Then $U(su(2)) \ltimes \mathcal{A}$ is the vector space $\mathcal{A} \otimes U(su(2))$, equipped with the structure of an associative algebra defined by $ua = (u_{(1)} \triangleright a)u_{(2)}$. Here $u_{(1)} \otimes u_{(2)}$ is the undeformed coproduct of $U(su(2))$.

In the following, we shall be interested in representations of $\mathcal{A}$ which have a "vacuum" vector $\rangle$ such that all elements can be written in the form $\mathcal{A}\rangle$, i.e. by acting with $\mathcal{A}$ on the vacuum vector. In particular, we will denote with $V_{\mathcal{A}}$ the free left $\mathcal{A}$–module $\mathcal{A}\rangle$ which as a vector space is equal to $\mathcal{A}$. This will be called the the "left vacuum representation" (or left regular representation). Now any[3] such representation of $\mathcal{A}$ can naturally be viewed as a representation of $U(su(2)) \ltimes \mathcal{A}$, if one declares the vacuum vector $\rangle$ to be a singlet under $U(su(2))$,

$$u\rangle = \varepsilon(u)\rangle,$$

and $u \triangleright (a\rangle) = (u \triangleright a)\rangle$. One can then verify the relations of $U(su(2)) \ltimes \mathcal{A}$.

Inspired by [123], we define for any $a \in \mathcal{A}$ the element

$$\hat{a} := (\mathcal{F}_1^{-1} \triangleright a)\mathcal{F}_2^{-1} \ \in U(su(2)) \ltimes \mathcal{A}. \tag{5-57}$$

Using the definition of the Drinfeld twist, it is immediate to verify the following properties:

$$\begin{aligned} \hat{a}\rangle &= a\rangle \\ \hat{a}\hat{b}\rangle &= (a \star b)\rangle \end{aligned} \tag{5-58}$$

where $(a \star b)$ is the twisted multiplication on $\mathcal{A}$ defined in (5-27). More generally,

$$\boxed{\hat{a}_1\hat{a}_2....\hat{a}_k\rangle = (a_1 \star (a_2 \star (....a_{k-1} \star a_k)...))\rangle} \tag{5-59}$$

for any $a_i \in \mathcal{A}$. Hence the elements $\hat{a}$ realize the twisted product (5-27) on $\mathcal{A}$, with this particular bracketing. If $c \in \mathcal{A}$ is a singlet, or equivalently $[c, U(su(2))] = 0$ in $U(su(2)) \ltimes \mathcal{A}$, then

$$\hat{c} = c. \tag{5-60}$$

If in addition the algebra $\mathcal{A}$ is commutative, then $\hat{c}$ is central in $U(su(2)) \ltimes \mathcal{A}$. Moreover, the new variables $\hat{a}_i$ are automatically covariant under the quantum group $U_q(su(2))$, with the q–deformed coproduct: denoting

$$\hat{u} := \varphi(u) \in U(su(2))$$

[3]provided the kernel of the representation is invariant under $U(su(2))$, which we shall assume.

for $u \in U_q(su(2))$, one easily verifies

$$\hat{u}\hat{a} = \widehat{u_1 \triangleright_q a}\; \widehat{u_2}, \tag{5-61}$$

where $u_1 \otimes u_2$ denotes the q–deformed coproduct. In particular,

$$\begin{aligned} \hat{u}\hat{a}\rangle &= u \triangleright_q a\rangle = \widehat{u \triangleright_q a}\rangle \\ \hat{u}\hat{a}\hat{b}\rangle &= (\widehat{u_1 \triangleright_q a})(\widehat{u_2 \triangleright_q b})\rangle. \end{aligned}$$

Therefore $U_q(su(2))$ acts correctly on the $\hat{a}$–variables in the left vacuum representation. More explicitly, assume that $\mathcal{A}$ is generated (as an algebra) by generators a_i transforming in the spin K representation π of $U(su(2))$, so that $ua_i = a_j \pi^j_i(u_{(1)})u_{(2)}$. Then (5-61) becomes

$$\hat{u}\hat{a}_i = \hat{a}_j \pi^j_i(\widehat{u}_1)\widehat{u}_2. \tag{5-62}$$

In general, the generators $\hat{a}_i$ will not satisfy closed commutation relations, even if the a_i do. However if $[a_i, a_j] = 0$, then one can verify that (cp. [123])

$$\hat{a}_i\hat{a}_j = \hat{a}_k\hat{a}_l\; \mathfrak{R}^{lk}_{ij} \tag{5-63}$$

where

$$\mathfrak{R}^{ij}_{kl} = (\pi^i_k \otimes \pi^j_l \otimes id)(\tilde{\phi}_{213}\; \tilde{R}_{12}\; \tilde{\phi}^{-1}_{123}) \quad \in U(su(2)). \tag{5-64}$$

Again, this involves only the coassociator and the universal $\mathcal{R}$–matrix. Such relations for field operators were already proposed in [80] on general grounds; here, they follow from the definition (5-57). In the case of several variables, one finds

$$\hat{a}_i\hat{b}_j = \hat{b}_k\hat{a}_l\; \mathfrak{R}^{lk}_{ij}. \tag{5-65}$$

Indeed, no closed quadratic commutation relations for deformed spaces of function with generators a_i in arbitrary representations of $U_q(su(2))$ are known, which has has been a major obstacle for defining QFT's on q–deformed spaces. In the present approach, the generators $\hat{a}_i$ satisfy quadratic commutation relations which close only in the bigger algebra $U(su(2)) \ltimes \mathcal{A}$. In general, they are not easy to work with. However some simplifications occur if we use minimal twists $\mathcal{F}$ as defined in Section 5.2, as was observed by Fiore [123]:

Proposition 5.4.1. *For minimal twists $\mathcal{F}$ as in (5-15), the following relation holds:*

$$g^{ij}\hat{a}_i\hat{a}_j = g^{ij}_c a_i a_j. \tag{5-66}$$

Here

$$g^{ij} = \pi^i_r(\mathcal{F}_1)\pi^j_s(\mathcal{F}_2)\; g^{rs}_c = g^{il}_c \pi^j_l(\gamma'), \tag{5-67}$$

where γ' is defined in (5-12). In particular if $\mathcal{A}$ is abelian, this implies that $g^{jk}\hat{a}_j\hat{a}_k$ is central in $U(su(2)) \ltimes \mathcal{A}$.

We include a short proof in Section 5.7 for convenience. This will be very useful to define a quantized field theory. From now on, we will always assume that the twists are minimal.

Derivatives. Let $\mathcal{A}$ be again the free commutative algebra with generators a_i in the spin K representation of $U(su(2))$, and consider the left vacuum representation $V_{\mathcal{A}} = \mathcal{A}\rangle$ of $U(su(2)) \ltimes \mathcal{A}$. Let ∂_i be the (classical) derivatives, which act as usual on the functions in $\mathcal{A}$. They can be considered as operators acting on $V_{\mathcal{A}}$, and as such they satisfy the relations of the classical Heisenberg algebra, $\partial_i a_j = g^c_{ij} + a_j \partial_i$. Now define

$$\hat{\partial}_i := (\mathcal{F}_1^{-1} \triangleright \partial_i)\mathcal{F}_2^{-1}, \tag{5-68}$$

which is an operator acting on $V_{\mathcal{A}} = \mathcal{A}\rangle$; in particular, it satisfies $\hat{\partial}_i\rangle = 0$. Then the following relations hold:

Proposition 5.4.2. *For minimal $\mathcal{F}$ as in (5-15), the operators $\hat{a}_i$, $\hat{\partial}_j$ acting on the left vacuum representation satisfy*

$$\begin{aligned} \hat{\partial}_i(g^{jk}\hat{a}_j\hat{a}_k) &= 2\hat{a}_i + (g^{jk}\hat{a}_j\hat{a}_k)\hat{\partial}_i, && (5\text{-}69)\\ \hat{\partial}_i\hat{a}_j &= g_{ij} + \hat{a}_k\hat{\partial}_l\, \mathfrak{R}^{lk}_{ij}. && (5\text{-}70) \end{aligned}$$

The proof is given in Section 5.7; the second relation (5-70) is again very close to a result (Proposition 6) in [123], and it holds in fact in $U(su(2)) \ltimes \mathcal{A}$. Of course, the brackets in (5-69) were just inserted for better readability, unlike in Lemma 5.3.2 where they were essential. If we have algebras with several variables in the same representation, then for example

$$\hat{\partial}_{a_i}(g^{jk}\hat{b}_j\hat{a}_k) = \hat{b}_i + (g^{jk}\hat{b}_j\hat{a}_k)\hat{\partial}_{a_i} \tag{5-71}$$

holds, in self–explanatory notation.

One advantage of this approach compared to the quasi–associative formalism in the previous section is that the concept of a star is clear, induced from Hilbert space theory. This will be explained next.

5.4.1 Reality structure.

Even though the results of this section are more general, we assume for simplicity that $\mathcal{A}$ is the free commutative algebra generated by the elements $\{a_i\}$ which transform in the spin K representation of $U(su(2))$ with integer K, i.e. the algebra of complex–valued functions on $\mathbb{R}^{2K+1}$ (or products thereof). Then the classical integral defines an invariant positive functional on $\mathcal{A}$ which satisfies (5-54), and $V_{\mathcal{A}}$ becomes a Hilbert space (5-55) (after factoring out a null space if necessary). Hence we can calculate the operator adjoint of the generators of this algebra. By construction,

$$a_i^* = g_c^{ij} a_j$$

where g_c^{ij} is the invariant tensor, normalized such that $g_c^{ij} g_c^{jk} = \delta^{ik}$. As discussed above, $V_{\mathcal{A}}$ is a representation of the semidirect product $U(su(2)) \ltimes \mathcal{A}$, in particular it is a unitary representation of $U(su(2))$. Hence the star on the generators of $U(su(2))$ is

$$H^* = H, \quad X^{\pm *} = X^{\mp}.$$

Now one can simply calculate the star of the twisted variables $\hat{a}_i \in U(su(2)) \ltimes \mathcal{A}$. The result is as expected:

Proposition 5.4.3. *If $\mathcal{F}$ is a minimal unitary twist as in Proposition 5.2.1, then the adjoint of the operator $\hat{a}_i$ acting on the left vacuum representation $\mathcal{A}\rangle$ is*

$$\hat{a}_i^* = g^{ij} \hat{a}_j. \tag{5-72}$$

This is proved in Section 5.7, and it was already found in [123]. It is straightforward to extend these results to the case of several variables $a_i^{(K)}, b_j^{(L)}, ...$ in different representations, using a common vacuum $\rangle$. The star structure is always of the form (5.4.3).

If $\mathcal{A}$ is the algebra of functions on $\mathbb{R}^{2K+1}$, we have seen above that the left vacuum representation $\mathcal{A}\rangle$ is also a representation of the Heisenberg algebra $\mathcal{A}_H$ with generators a_i, ∂_j. Again we can calculate the operator adjoints, and the result is

$$\begin{aligned} \partial_i^* &= -g_c^{ij} \partial_j, \\ \hat{\partial}_i^* &= -g^{ij} \hat{\partial}_j. \end{aligned}$$

Of course, all these statements are on a formal level, ignoring operator–technical subtleties.

5.4.2 Relation with the quasiassociative $\star$ –product

Finally, we make a simple but useful observation, which provides the connection of the operator approach in this section with the quasiassociative approach of Section 5.3. Observe first that an invariant (real, positive (5-54)) functional $\langle\ \rangle$ on $\mathcal{A}$ extends trivially as a (real, positive) functional on $U(su(2)) \ltimes \mathcal{A}$, by evaluating the generators of $U(su(2))$ on the left (or right) of $\mathcal{A}$ with the counit. Now for any tensor $I^{i_1 \dots i_k}$ of $U_q(su(2))$, denote

$$I(\hat{a}) := I^{i_1 \dots i_k} \hat{a}_{i_1} ... \hat{a}_{i_k} \qquad \in U(su(2)) \ltimes \mathcal{A},$$

and

$$I_\star(a) := I^{i_1 \dots i_k} a_{i_1} \star (... \star (a_{i_{k-1}} \star a_{i_k})...) \qquad \in \mathcal{A}. \tag{5-73}$$

Then the following holds:

Lemma 5.4.4. *1) If $I = I^{i_1 \dots i_k}$ is an invariant tensor of $U_q(su(2))$, then $I(\hat{a})$ as defined above commutes with $u \in U_q(su(2))$,*

$$[u, I(\hat{a})] = 0 \qquad \textit{in} \;\; U(su(2)) \ltimes \mathcal{A}. \tag{5-74}$$

2) Let $s\rangle \in \mathcal{A}\rangle$ *be invariant, i.e.* $u \cdot s\rangle = \varepsilon_q(u)s\rangle$, *and* I, ..., J *be invariant tensors of* $U_q(su(2))$. *Then*

$$I(\hat{a})...J(\hat{a})\ s\rangle = I_\star(a) \star .... \star J_\star(a)\ s\rangle.$$

3) Let I, J *be invariant, and* $P = P^{i_1...i_k}$ *be an arbitrary tensor of* $U_q(su(2))$. *Denote with* P_0 *the trivial component of* P *under the action of* $U_q(su(2))$. *Then for any invariant functional* $\langle\ \rangle$ *on* $\mathcal{A}$,

$$\begin{aligned} \langle I(\hat{a})...J(\hat{a})\ P(\hat{a})\rangle &= \langle I(\hat{a})...J(\hat{a})\ P_0(\hat{a})\rangle \\ &= \langle I_\star(a) \star .... \star J_\star(a)(P_0)_\star(a)\rangle \\ &= \langle I_\star(a) \star .... \star J_\star(a) \star P_\star(a)\rangle \end{aligned} \tag{5-75}$$

Moreover if $\mathcal{A}$ *is abelian, then the* $I(\hat{a})$, $J(\hat{a})$ *etc. can be considered as central in an expression of this form, for example*

$$\langle I(\hat{a})...J(\hat{a})\ P(\hat{a}_i)\rangle = \langle P(\hat{a}_i)\ I(\hat{a})...J(\hat{a})\rangle = \langle P(\hat{a}_i)\ J(\hat{a})...I(\hat{a})\rangle$$

and so on.

The proof follows easily from (5-61), (5-59) and Lemma 5.3.1. The stars between the invariant polynomials $I_\star(a), ..., J_\star(a)$ are of course trivial, and no brackets are needed.

5.5 Twisted Euclidean QFT

These tools can now be applied to our problem of quantizing fields on the q–deformed fuzzy sphere $\mathcal{S}^2_{q,N}$. Most of the discussion is not restricted to this space, but it is on a much more rigorous level there because the number of modes is finite. We will present 2 approaches, the first based on twisted $\star$ –products as defined in Section 5.3, and the second using an operator formalism as in Section 5.4. Both have their own merits which seem to justify presenting them both. Their equivalence will follow from Lemma 5.4.4.

First, we discuss some basic requirements for a quantum field theory on spaces with quantum group symmetry. Consider a scalar field, and expand it in its modes as

$$\Psi(x) = \sum_{K,n} \psi_{K,n}(x)\ a^{K,n}. \tag{5-76}$$

Here the $\psi_{K,n}(x) \in \mathcal{S}^2_{q,N}$ are a basis of the spin K representation of $U_q(su(2))$,

$$u \triangleright_q \psi_{K,n}(x) = \psi_{K,m}(x)\pi^m_n(u), \tag{5-77}$$

and the coefficients $a^{K,n}$ transform in the dual (contragredient) representation of $\tilde{U}_q(su(2))$,

$$u\ \tilde{\triangleright}_q a^{K,n} = \pi^n_m(\tilde{S}u)\ a^{K,m}. \tag{5-78}$$

It is important to distinguish the Hopf algebras which act on the coefficients $a^{K,n}$ and on the functions $\psi_{K,n}(x)$, respectively. The Hopf algebra $\tilde{U}_q(su(2))$ is obtained from $U_q(su(2))$ by flipping the coproduct and using the opposite antipode $\tilde{S} = S^{-1}$. In particular, the $\mathcal{R}$–matrix and the invariant tensors are also flipped:

$$\tilde{g}^K_{nm} = g^K_{mn} \tag{5-79}$$

where $\tilde{g}^K_{nm}$ is the invariant tensor of $\tilde{U}_q(su(2))$. The reason for this will become clear soon. Moreover, it is sometimes convenient to express the contragredient generators in terms of "ordinary" ones,

$$a_{K,n} = \tilde{g}^K_{nm} a^{K,m}. \tag{5-80}$$

Then $\tilde{U}_q(su(2))$ acts as

$$u \,\tilde{\triangleright}_q a_{K,n} = a_{K,m}\ \pi^m_n(u).$$

We assume that the coefficients $a^{K,n}$ generate some algebra $\mathcal{A}$. This is not necessarily the algebra of field operators, which in fact would not be appropriate in the Euclidean case even for $q = 1$. Rather, $\mathcal{A}$ could be the algebra of coordinate functions on configuration space (space of modes) for $q = 1$, and an analog thereof for $q \neq 1$. The fields $\Psi(x)$ can then be viewed as "algebra–valued distributions" in analogy to usual field theory, by defining

$$\Psi[f] := \int\limits_{\mathcal{S}^2_{q,N}} \Psi(x) f(x) \quad \in \mathcal{A}$$

for $f(x) \in \mathcal{S}^2_{q,N}$. Then the covariance properties (5-77) and (5-78) could be stated as

$$u \,\tilde{\triangleright}_q \Psi[f] = \Psi[u \triangleright_q f], \tag{5-81}$$

using the fact that $\int (u \triangleright_q f) g = \int f(S(u) \triangleright g)$.

Our goal is to define some kind of correlation functions of the form

$$\boxed{\langle \Psi[f_1]\Psi[f_2]...\Psi[f_k]\rangle \quad \in \mathbb{C}} \tag{5-82}$$

for any $f_1, ..., f_k \in \mathcal{S}^2_{q,N}$, in analogy to the undeformed case. After "Fourier transformation" (5-76), this amounts to defining objects

$$G^{K_1,n_1;K_2,n_2;...;K_k,n_k} := \langle a^{K_1,n_1} a^{K_2,n_2} ... a^{K_k,n_k}\rangle =: \langle P(a)\rangle \tag{5-83}$$

where $P(a)$ will denote some polynomial in the $a^{K,n}$ from now on, perhaps by some kind of a "path integral" $\langle P(a)\rangle = \frac{1}{\mathcal{N}} \int \Delta a\ e^{-S[\Psi]} P(a)$. We require that they should satisfy at least the following properties, to be made more precise later:

(1) *Covariance:*

$$\langle u \,\tilde{\triangleright}_q P(a)\rangle = \varepsilon_q(u)\ \langle P(a)\rangle, \tag{5-84}$$

which means that the $G^{K_1,n_1;K_2,n_2;...;K_k,n_k}$ are invariant tensors of $\tilde{U}_q(su(2))$,

(2) *Hermiticity:*

$$\langle P(a)\rangle^* = \langle P^*(a)\rangle \tag{5-85}$$

for a suitable involution $*$ on $\mathcal{A}$,

(3) *Positivity:*

$$\langle P(a)^* P(a)\rangle \geq 0, \tag{5-86}$$

(4) *Symmetry*

under permutations of the fields, in a suitable sense discussed below.

This will be our heuristic "working definition" of a quantum group covariant Euclidean QFT.

In particular, the word "symmetry" in (4) needs some explanation. The main purpose of a symmetrization axiom is that it puts a restriction on the number of degrees of freedom in the model, which in the limit $q \to 1$ should agree with the undeformed case. More precisely, the amount of information contained in the correlation functions (5-83) should be the same as for $q = 1$, i.e. the Poincare series of $\mathcal{A}$ should be the same. This means that the polynomials in the $a^{K,n}$ can be ordered as usual, i.e. they satisfy some kind of "Poincare–Birkhoff–Witt" property. This is what we mean with "symmetry" in (4). In more physical terms, it implies the statistical properties of bosons[4].

However, it is far from trivial how to impose such a "symmetry" on tensors which are invariant under a quantum group. Ordinary symmetry is certainly not consistent with covariance under a quantum group. One might be tempted to replace "symmetry" by some kind of invariance under the braid group which is naturally associated to any quantum group. This group is generally much bigger than the group of permutations, however, and such a requirement is qualitatively different and leaves fewer degrees of freedom. The properties (3) and (4) are indeed very nontrivial requirements for a QFT with a quantum group spacetime symmetry, and they are not satisfied in the proposals that have been given up to now, to the knowledge of the authors.

Covariance (1) suggests that the algebra $\mathcal{A}$ generated by the $a^{K,n}$ is a $U_q(su(2))$–module algebra. This implies immediately that $\mathcal{A}$ cannot be commutative, because the coproduct of $U_q(su(2))$ is not cocommutative. The same conclusion can be reached by contemplating the meaning of invariance of an action $S[\Psi]$, which will be clarified below. One could even say that a second quantization is required by consistency. As a further guiding line, the above axioms (1) – (4) should be verified easily in a "free" field theory.

In general, there is no obvious candidate for an associative algebra $\mathcal{A}$ satisfying all these requirements. We will construct a suitable quasiassociative algebra $\mathcal{A}$ as a star–deformation of the algebra of functions on configuration space along the lines of Section 5.3, which satisfies these requirements. Our approach is rather general and should be applicable in a more general context, such as for higher–dimensional theories. Quasiassociativity implies that the correlation functions (5-82) make sense only after specifying the order in which the fields should be multiplied (by explicitly putting brackets), however different ways of bracketing are always related

[4] we do not consider fermions here.

by a unitary transformation. Moreover, the correct number of degrees of freedom is guaranteed by construction. We will then define QFT's which satisfy the above requirements using a path integral over the fields $\Psi(x)$, i.e. over the modes $a^{K,n}$. An associative approach will also be presented in Section 5.5.2, which is essentially equivalent.

5.5.1 Star product approach

The essential step is as follows. Using the map φ (5-3), the coefficients $a^{K,n}$ transform also under the spin K representation of $U(su(2))$, via $u \;\tilde{\triangleright} a^{K,n} = \varphi^{-1}(u) \;\tilde{\triangleright}_q a^{K,n}$. Hence we can consider the usual commutative algebra $\mathcal{A}^K$ of functions on $\mathbb{R}^{2K+1}$ generated by the $a^{K,n}$, and view it as a left $U(su(2))$–module algebra $(\mathcal{A}^K, \cdot, \tilde{\triangleright})$. As explained in the Section 5.3, we can then obtain from it the left $\tilde{U}_q(su(2))$–module algebra $(\mathcal{A}^K, \star, \tilde{\triangleright}_q)$, with multiplication $\star$ as defined in (5-27). More generally, we consider the left $\tilde{U}_q(su(2))$–module algebra $(\mathcal{A}, \star, \tilde{\triangleright}_q)$ where $\mathcal{A} = \otimes_{K=0}^N \mathcal{A}^K$. Notice that the twist $\tilde{\mathcal{F}}$ corresponding to the reversed coproduct must be used here, which is simply $\tilde{\mathcal{F}}_{12} = \mathcal{F}_{21}$. The reality issues will be discussed in Section 5.5.2.

Invariant actions. Consider the following candidate for an invariant action,

$$S_{int}[\Psi] = \int\limits_{\mathcal{S}^2_{q,N}} \Psi(x) \star (\Psi(x) \star \Psi(x)). \tag{5-87}$$

Assuming that the functions on $\mathcal{S}^2_{q,N}$ commute with the coefficients, $[x_i, a^{K,n}] = 0$, this can be written as

$$\begin{aligned} S_{int}[\Psi] &= \int\limits_{\mathcal{S}^2_{q,N}} \psi_{K,n}(x)\; \psi_{K',m}(x)\; \psi_{K'',l}(x)\; a^{K,n} \star (a^{K',m} \star a^{K'',l}) \\ &= I^{(3)}_{K,K',K'';\,n,m,l}\; a^{K,n} \star (a^{K',m} \star a^{K'',l}) \quad \in \mathcal{A}. \end{aligned} \tag{5-88}$$

Here[5] $I^{(3)}_{K,K',K'';n,m,l} = \int\limits_{\mathcal{S}^2_{q,N}} \psi_{K,n}\; \psi_{K',m}\; \psi_{K'',l}$ is by construction an invariant tensor of $U_q(su(2))$,

$$I^{(3)}_{K,K',K'';n,m,l}\; \pi^n_r(u_1)\pi^m_s(u_2)\pi^l_t(u_3) = \varepsilon_q(u)\; I^{(3)}_{K,K',K'';r,s,t}. \tag{5-89}$$

We have omitted the labels on the various representations. Hence $S_{int}[\Psi]$ is indeed an invariant element of $\mathcal{A}$:

$$\begin{aligned} u \;\tilde{\triangleright}_q S_{int}[\Psi] &= I^{(3)}_{K,K',K'';\,n,m,l}\; \pi^n_r(\tilde{S}u_{\tilde{1}})\pi^m_s(\tilde{S}u_{\tilde{2}})\pi^l_t(\tilde{S}u_{\tilde{3}})\; a^{K,r} \star (a^{K',s} \star a^{K'',t}) \\ &= \varepsilon_q(u)\; S_{int}[\Psi] \end{aligned}$$

[5] note that the brackets are actually not necessary here because of (5-23). For higher–order terms they are essential, however.

using (5-89), where $u_{\tilde{1}} \otimes u_{\tilde{2}} \otimes u_{\tilde{3}}$ is the 2–fold coproduct of $u \in \tilde{U}_q(su(2))$; notice that the antipode reverses the coproduct. This is the reason for using $\tilde{U}_q(su(2))$.

In general, our actions $S[\Psi]$ will be polynomials in $\mathcal{A}$, and we shall only consider invariant actions,

$$u \,\tilde{\triangleright}_q S[\Psi] = \varepsilon_q(u)\; S[\Psi] \qquad \in \mathcal{A}, \tag{5-90}$$

for $u \in \tilde{U}_q(su(2))$. It is important to note that by Lemma 5.3.1, the star product of any such invariant actions is commutative and associative, even though the full algebra of the coefficients $(\mathcal{A}, \star)$ is not. Moreover we only consider actions which are obtained using an integral over $\mathcal{S}^2_{q,N}$ as in (5-87), which we shall refer to as "local".

In particular, consider the quadratic action

$$S_2[\Psi] = \int\limits_{\mathcal{S}^2_{q,N}} \Psi(x) \star \Psi(x),$$

which can be rewritten as

$$\begin{aligned} S_2[\Psi] &= \int\limits_{\mathcal{S}^2_{q,N}} \psi_{K,n}(x)\; \psi_{K,m}(x)\; a^{K,n} \star a^{K,m} = \sum_{K=0}^{N} g^K_{nm}\; a^{K,n} \star a^{K,m} \\ &= \sum_{K=0}^{N} \tilde{g}^K_{mn}\; a^{K,n} \star a^{K,m}. \end{aligned} \tag{5-91}$$

Here we assumed that the basis $\psi_{K,n}(x)$ is normalized such that

$$\int\limits_{\mathcal{S}^2_{q,N}} \psi_{K,n}(x)\psi_{K',m}(x) = \delta_{K,K'}\; g^K_{n,m}. \tag{5-92}$$

This action is of course invariant, $u \,\tilde{\triangleright}_q S_2[\Psi] = \varepsilon_q(u)\; S_2[\Psi]$. Moreover, the invariant quadratic actions agree precisely with the classical ones. Indeed, the most general invariant quadratic action has the form

$$\begin{aligned} S_{free}[\Psi] &= \sum_{K=0}^{N} D_K g^K_{nm}\; a^{K,n} \star a^{K,m} \\ &= \sum_{K=0}^{N} D_K (g^K_c)_{nm}\; a^{K,n} \cdot a^{K,m} \end{aligned} \tag{5-93}$$

using (5-46), for some $D_K \in \mathbb{C}$. This will allow to derive Feynman rules from Gaussian integrals as usual.

Quantization: path integral. We will define the quantization by a (configuration space) path integral, i.e. some kind of integration over the possible values of the coefficients $a^{K,m}$. This integral should be invariant under $\tilde{U}_q(su(2))$. Following Section 5.3.1, we consider $\mathcal{A}^K$ as the vector space of complex–valued functions on $\mathbb{R}^{2K+1}$, and use the usual classical integral over $\mathbb{R}^{2K+1}$. Recall that the algebra structure of $\mathcal{A}^K$ does not enter here at all. The same approach was used in [61] to define the quantization of the undeformed fuzzy sphere, and an analogous approach is usually taken on spaces with a star product [5]. Notice that K is an integer, since we do not consider fermionic fields here. Explicitly, let $\int d^{2K+1}a^K \;\; f$ be the integral of an element $f \in \mathcal{A}^K$ over $\mathbb{R}^{2K+1}$. It is invariant under the action of $\tilde{U}_q(su(2))$ (or equivalently under $U(su(2))$) as discussed in Section 5.3.1:

$$\int d^{2K+1}a^K \; u \,\tilde{\triangleright}_q f = \varepsilon_q(u) \int d^{2K+1}a^K \; f.$$

Now we define

$$\int \mathcal{D}\Psi \; f[\Psi] := \int \prod_K d^{2K+1}a^K \; f[\Psi],$$

where $f[\Psi] \in \mathcal{A}$ is any integrable function (in the usual sense) of the variables $a^{K,m}$. This will be our path integral, which is by construction invariant under the action $\tilde{\triangleright}_q$ of $\tilde{U}_q(su(2))$.

Correlation functions can now be defined as functionals of "bracketed polynomials" $P_\star(a) = a^{K_1,n_1} \star (a^{K_2,n_2} \star (... \star a^{K_l,n_l}))$ in the field coefficients by

$$\boxed{\langle P_\star(a)\rangle := \frac{\int \mathcal{D}\Psi \; e^{-S[\Psi]} P_\star(a)}{\int \mathcal{D}\Psi \; e^{-S[\Psi]}}.} \tag{5-94}$$

This is natural, because all invariant actions $S[\Psi]$ commute with the generators $a^{K,n}$. Strictly speaking there should be a factor $\frac{1}{\hbar}$ in front of the action, which we shall omit. In fact there are now 3 different "quantization" parameters: $\hbar$ has the usual meaning, while N and $q - q^{-1}$ determines a quantization or deformation of space.

Invariance of the action $S[\Psi] \in \mathcal{A}$ implies that

$$\langle u \,\tilde{\triangleright}_q P_\star(a)\rangle = \varepsilon_q(u) \; \langle P_\star(a)\rangle, \tag{5-95}$$

and therefore

$$\langle P_\star(a)\rangle = \langle (P_0)_\star(a)\rangle \tag{5-96}$$

where P_0 is the singlet part of the polynomial P, as in Lemma 5.4.4. These are the desired invariance properties, and they would not hold if the $a^{K,n}$ were commuting variables. By construction, the number of independent modes of a polynomial $P_\star(a)$ with given degree is the same as for $q = 1$. One can in fact order them, using quasiassociativity together with the commutation relations (5-35) which of course also hold under the integral:

$$\langle P_\star(a) \star ((a_i \star a_j - a_k \star a_l \; \tilde{\mathcal{R}}^{lk}_{ij}) \star Q_\star(a))\rangle = 0, \tag{5-97}$$

for any polynomials $P_\star(a), Q_\star(a) \in \mathcal{A}$. This can also be verified using the perturbative formula (5-102) below. Therefore the symmetry requirement (4) of Section 5.5 is satisfied. Moreover, the following cyclic property holds:

$$\langle a_i \star P_\star(a)\rangle = \langle P_\star(a) \star a_k\rangle \; \tilde{D}^k_i, \qquad \tilde{D}^k_i = \tilde{g}^{kn}\tilde{g}_{in} \tag{5-98}$$

for any $P_\star(a)$. This follows using (5-96) and the well–known cyclic property of the q–deformed invariant tensor $\tilde{g}_{ij}$.

In general, the use of quasiassociative algebras for QFT is less radical than one might think, and it is consistent with results of [8] on boundary correlation functions in BCFT. Before addressing the issue of reality, we develop some tools to actually calculate such correlation functions in perturbation theory.

Currents and generating functionals. One can now introduce the usual tools of quantum field theory. We introduce (external) currents $J(x)$ by

$$J(x) = \sum_{K,n} \psi_{K,n}(x) \; j^{K,n}, \tag{5-99}$$

where the new generators $j^{K,n}$ are included into the $\tilde{U}_q(su(2))$–module algebra $\mathcal{A}$, again by the twisted product (5-27). We can then define a generating functional

$$\boxed{Z[J] = \tfrac{1}{\mathcal{N}} \int \mathcal{D}\Psi \; e^{-S[\Psi] + \int \Psi(x)\star J(x)}}, \tag{5-100}$$

which is an element of $\mathcal{A}$ but depends only on the current variables. Here $\mathcal{N} = \int \mathcal{D}\Psi \; e^{-S[\Psi]}$. Note that

$$\int \Psi(x) \star J(x) = \int J(x) \star \Psi(x),$$

which follows e.g. from (5-93). Invariance of the functional integral implies that

$$u \,\tilde{\triangleright}_q Z[J] = \varepsilon_q(u) \; Z[J] \tag{5-101}$$

for any $u \in \tilde{U}_q(su(2))$, provided the actions $S[\Psi]$ are invariant.

It is now useful to introduce derivatives $\partial^{K,n}_{(j)}$ similar to (5-43), which together with the currents form a twisted (quasiassociative) Heisenberg algebra as explained in the previous section:

$$\partial^K_{(j)n} \star j^{K'}_m = \delta_{K,K'} \; \tilde{g}^K_{nm} + \; j^{K'}_r \star \partial^K_{(j)s} \; \tilde{\mathcal{R}}^{sr}_{nm}$$

By a calculation analogous to (5-49), it follows that

$$\partial^{K,n}_{(j)} \Big(\int \Psi(x) \star J(x) \Big) = a^{K,n} + \Big(\int \Psi(x) \star J(x) \Big) \partial^{K,n}_{(j)}.$$

Recall that it is not necessary to put a star if one of the factors is a singlet.

This is exactly what we need. We conclude immediately that $[\partial_{(j)}^{K,n}, \exp(\int \Psi\star J)] = a^{K,n}\exp(\int \Psi\star J)$, and by an inductive argument it follows that the correlation functions (5-94) can be written as

$$\langle P_\star(a)\rangle = {}_{J=0}\langle P_\star(\partial_{(j)})\ Z[J]\rangle_{\partial=0}. \tag{5-102}$$

Here ${}_{J=0}\langle ...\rangle_{\partial=0}$ means ordering the derivatives to the right of the currents and *then* setting J and $\partial_{(j)}$ to zero. The substitution of derivatives into the bracketed polynomial $P_\star$ is well–defined, because the algebra of the generators a is the same as the algebra of the derivatives $\partial_{(j)}$.

The usual perturbative expansion can now be obtained easily. Consider a quadratic action of the form

$$S_{free}[\Psi] = \int\limits_{\mathcal{S}^2_{q,N}} \Psi(x)\star D\Psi(x),$$

where D is an invariant (e.g. differential) operator on $\mathcal{S}^2_{q,N}$, so that $D\Psi(x) = \sum \psi_{K,n}(x)\ D_K a^{K,n}$ with $D_K \in \mathbb{C}$. It then follows as usual that

$$Z_{free}[J] := \frac{1}{\mathcal{N}_{free}} \int \mathcal{D}\Psi\ e^{-S_{free}[\Psi]+\int \Psi(x)\star J(x)} = e^{\frac{1}{2}\int J(x)\star D^{-1}J(x)}. \tag{5-103}$$

This implies that after writing the full action in the form $S[\Psi] = S_{free}[\Psi] + S_{int}[\Psi]$, one has

$$\begin{aligned} Z[J] &= \frac{1}{\mathcal{N}} \int \mathcal{D}\Psi\ e^{-S_{int}[\Psi]} e^{-S_{free}[\Psi]+\int \Psi(x)\star J(x)} \\ &= \frac{1}{\mathcal{N}'}\ e^{-S_{int}[\partial_{(j)}]}\ Z_{free}[J]\rangle_{\partial=0}. \end{aligned} \tag{5-104}$$

This is the starting point for a perturbative evaluation. In the next section, we shall cast this into a form which is even more useful, and show that the "vacuum diagrams" cancel as usual.

Relation with the undeformed case. There is a conceptually simple relation of all the above models which are invariant under $\tilde{U}_q(su(2))$ with models on the undeformed fuzzy sphere which are invariant under $U(su(2))$, at the expense of "locality". First, note that the space of invariant actions (5-90) is independent of q. More explicitly, consider an interaction term of the form (5-88). If we write down explicitly the definition of the $\star$ product of the $a^{K,n}$ variables, then it can be viewed as an interaction term of $a^{K,n}$ variables with a tensor which is invariant under the *undeformed* $U(su(2))$, obtained from $I^{(3)}_{K,K',K'';\,n,m,l}$ by multiplication with representations of $\mathcal{F}$. In the limit $q=1$, this $\mathcal{F}$ becomes trivial. In other words, the above actions can also be viewed as actions on undeformed fuzzy sphere $S^2_{q=1,N}$, with interactions which are "nonlocal" in the sense of $S^2_{q=1,N}$, i.e. they are given by traces of products of matrices only to the lowest order in $(q-1)$. Upon spelling out the $\star$ product in the correlation functions (5-94) as well, they can be considered as ordinary correlation functions of a slightly nonlocal field theory on $S^2_{q=1,N}$, disguised by the transformation $\mathcal{F}$.

In this sense, q–deformation simply amounts to some kind of nonlocality of the interactions. A similar interpretation is well–known in the context of field theories on spaces with a Moyal

product [5]. The important point is, however, that one can calculate the correlation functions for $q \neq 1$ *without* using the twist $\mathcal{F}$ explicitly, using only $\hat{R}$ –matrices and the coassociators $\tilde{\phi}$, which are much easier to work with. This should make the q–deformed point of view useful. It is also possible to generalize these results to other q–deformed spaces.

5.5.2 Associative approach

In order to establish the reality properties of the field theories introduced above, it is easier to use an alternative formulation, using the results of Section 5.4. The equivalence of the two formulations will follow from Section 5.4.2. This will also allow to define field operators for second–quantized models in 2+1 dimensions in Section 5.6.4.

Consider the left vacuum representations $V_{\mathcal{A}} = \mathcal{A}\rangle$ of $\mathcal{A} = \otimes_K \mathcal{A}^K$ introduced in Section 5.4, and define the operators[6]

$$\hat{a}^{K,n} = (\tilde{\mathcal{F}}_1^{-1} \triangleright a^{K,n})\tilde{\mathcal{F}}_2^{-1} \quad \in \; U(su(2)) \ltimes \mathcal{A} \tag{5-105}$$

acting on $\mathcal{A}\rangle$. We can then more or less repeat all the constructions of the previous section with $a^{k,n}$ replaced by $\hat{a}^{K,n}$, omitting the $\star$ product. The covariance property (5-81) of the field

$$\hat{\Psi}(x) = \sum_{K,n} \psi_{K,n}(x) \; \hat{a}^{K,n} \tag{5-106}$$

can now be written in the form

$$\Psi[u \triangleright_q f] = u_{\tilde{1}} \Psi[f] \tilde{S} u_{\tilde{2}}.$$

Invariant actions can be obtained by contracting the $\hat{a}^{K,n}$ with invariant tensors of $\tilde{U}_q(su(2))$, and satisfy

$$[u, S[\hat{\Psi}]] = 0$$

for[7] $u \in \tilde{U}_q(su(2))$. For example, any actions of the form

$$S[\hat{\Psi}] = \int\limits_{\mathcal{S}^2_{q,N}} \hat{\Psi}(x) D \hat{\Psi}(x) + \lambda \hat{\Psi}(x)\hat{\Psi}(x)\hat{\Psi}(x) \;= S_{free}[\hat{\Psi}] + S_{int}[\hat{\Psi}] \qquad \in \; U(su(2)) \ltimes \mathcal{A}$$

are invariant, where D is defined as before. Using Proposition 5.4.1, the quadratic invariant actions again coincide with the undeformed ones. In general, higher–order actions are elements of $U(su(2)) \ltimes \mathcal{A}$ but not of $\mathcal{A}$. Nevertheless as explained in Section 5.4.2, all such invariant actions $S[\hat{\Psi}]$ are in one–to–one correspondence with invariant actions in the $\star$–product approach, with brackets as in (5-73). This will be understood from now on.

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[6] they should not be considered as field operators.

[7] recall that as algebra, there is no difference between $U(su(2))$ and $\tilde{U}_q(su(2))$.

Consider again the obvious (classical) functional $\int \prod_K d^{2K+1} a^K$ on $\mathcal{A}$ (or $V_{\mathcal{A}}$) as in the previous section, and recall from Section 5.4.2 that it extends trivially to a functional on $U(su(2)) \ltimes \mathcal{A}$, by evaluating $U(su(2))$ with ε. We will denote this functional by $\int \mathcal{D}\hat{\Psi}$. Define correlation functions of polynomials in the $\hat{a}^{K,n}$ variables as

$$\langle P(\hat{a})\rangle := \frac{\int \mathcal{D}\hat{\Psi} e^{-S[\hat{\Psi}]} P(\hat{a})}{\int \mathcal{D}\hat{\Psi} e^{-S[\hat{\Psi}]}} = \langle P_0(\hat{a})\rangle. \tag{5-107}$$

Here P_0 is again the singlet part of the polynomial P. Then Lemma 5.4.4 implies

$$\langle P(\hat{a})\rangle = \langle P_\star(a)\rangle, \tag{5-108}$$

always assuming that the actions $S[\hat{\Psi}]$ are invariant under $U_q(su(2))$. This shows the equivalence with the approach of the previous section. Moreover,

$$\langle P(\hat{a})\ \hat{a}_i \hat{a}_j\ Q(\hat{a})\rangle = \langle P(\hat{a})\ \hat{a}_k \hat{a}_l\ \mathfrak{R}^{lk}_{ij}\ Q(\hat{a})\rangle, \tag{5-109}$$

follows from (5-63), or from (5-112) below on the perturbative level.

Currents and generating functionals. We can again extend $\mathcal{A}$ by other variables such as currents

$$\hat{J}(x) = \sum_{K,n} \psi_{K,n}(x)\ \hat{j}^{K,n} \qquad \in U(su(2)) \ltimes \mathcal{A}, \tag{5-110}$$

and consider the generating functional

$$Z[\hat{J}] = \frac{1}{\mathcal{N}} \int \mathcal{D}\hat{\Psi}\ e^{-S[\hat{\Psi}] + \int \hat{\Psi}(x)\hat{J}(x)}\rangle \tag{5-111}$$

with $Z[0] = 1$. This is defined as the element of $\mathcal{A}\rangle \cong \mathcal{A}$ obtained after integrating over the a^K–variables; the result depends on the currents only. The brace $\rangle$ indicates that the explicit $U(su(2))$ factors in $U(su(2)) \ltimes \mathcal{A}$ are evaluated by ε. Again, Lemma 5.4.4 implies that $Z[\hat{J}]$ agrees precisely with the previous definition (5-100).

As explained in Section 5.4, one can consider also the twisted derivative operators $\hat{\partial}^{K,n}_{(j)}$, which act on $\mathcal{A}\rangle$. Using Proposition 5.4.2, we can derive essentially the same formulas as in the previous section, omitting the star product. In particular, (5-71) implies that

$$\hat{\partial}^{K,n}_{(j)}\Big(\int \hat{\Psi}(x)\hat{J}(x)\Big) = \hat{a}^{K,n} + \Big(\int \hat{\Psi}(x)\hat{J}(x)\Big)\hat{\partial}^{K,n}_{(j)}.$$

Since invariant elements of $\mathcal{A}$ are central as was pointed out below (5-60), we obtain as usual

$$\langle P(\hat{a})\rangle = {}_{J=0}\langle P(\hat{\partial}_{(j)})\ Z[\hat{J}]\rangle_{\partial=0} \tag{5-112}$$

$$Z[\hat{J}] = \frac{1}{\mathcal{N}} \int \mathcal{D}\hat{\Psi}\ e^{-(S_{free}[\hat{\Psi}] + S_{int}[\hat{\Psi}]) + \int \hat{\Psi}(x)\hat{J}(x)}\rangle = \frac{1}{\mathcal{N}'}\ e^{-S_{int}[\hat{\partial}_{(j)}]}\ Z_{free}[\hat{J}]\rangle_{\partial=0}$$

$$Z_{free}[\hat{J}] = \frac{1}{\mathcal{N}_{free}} \int \mathcal{D}\hat{\Psi}\ e^{-S_{free}[\hat{\Psi}] + \int \hat{\Psi}(x)\hat{J}(x)}\rangle = e^{\frac{1}{2}\int \hat{J}(x) D^{-1} \hat{J}(x)}\rangle. \tag{5-113}$$

Even though these formulas can be used to calculate correlators perturbatively, there is a form which is more convenient for such calculations. To derive it, observe that (5-69) implies

$$\hat{\partial}^{K,n}_{(j)}\; e^{\frac{1}{2}\int \hat{J}(x)D^{-1}\hat{J}(x)} = e^{\frac{1}{2}\int \hat{J}(x)D^{-1}\hat{J}(x)}\; (D^{-1}_K\;\hat{j}^{K,n} + \hat{\partial}^{K,n}_{(j)}); \tag{5-114}$$

one can indeed verify that the algebra of

$$\hat{b}^{K,n} = D^{-1}_K\;\hat{j}^{K,n} + \hat{\partial}^{K,n}_{(j)} \tag{5-115}$$

is the same as the algebra of $\hat{a}^{K,n}$. Therefore (5-112) can be rewritten as

$$\begin{aligned} \langle P(\hat{a})\rangle &= \frac{1}{\mathcal{N}'}\;{}_{J=0}\Big\langle P(\hat{\partial}_{(j)})\; e^{-S_{int}[\hat{\partial}_{(j)}]}\; e^{\frac{1}{2}\int \hat{J}(x)D^{-1}\hat{J}(x)}\Big\rangle_{\partial=0} \\ &= \frac{1}{\mathcal{N}'}\;{}_{J=0}\langle e^{\frac{1}{2}\int \hat{J}D^{-1}\hat{J}} P(\hat{b})\; e^{-S_{int}[\hat{b}]}\rangle_{\partial=0} \\ &= \frac{{}_{J=0}\langle P(\hat{b})\; e^{-S_{int}[\hat{b}]}\rangle_{\partial=0}}{{}_{J=0}\langle e^{-S_{int}[\hat{b}]}\rangle_{\partial=0}}. \end{aligned} \tag{5-116}$$

To evaluate this, one reinserts the definition (5-115) of $\hat{b}$ as a sum of derivative operators $\hat{\partial}$ and current generators $\hat{j}$. Each $\hat{\partial}$ must be "contracted" with a $\hat{j}$ to the right of it using the commutation relations (5-70), which gives the inverse propagator D^{-1}_K, and the result is the sum of all possible complete contractions. This is the analog of Wick's theorem. The contractions can be indicated as usual by pairing up the $\hat{b}$ variables with a line, before actually reordering them. Then each contribution can be reconstructed uniquely from a given complete contraction; this could be stated in terms of Feynman rules.

One can also show that the denominator exactly cancels the "vacuum bubbles" in the numerator, as usual. Indeed, consider any given complete contraction of a term

$$\hat{b}...\hat{b}\;\frac{1}{n!}(S_{int}[\hat{b}])^n.$$

Mark the set of vertices which are connected (via a series of contractions) to some of the explicit $\hat{b}$ generators on the left with blue, and the others with red. Then 2 neighboring red and blue vertices can be interchanged keeping the given contractions, without changing the result. This is because only the homogeneous part of the commutation relations[8] (5-70) applies, and all vertices are singlets (cp. Lemma 5.4.4). Therefore the red vertices can be moved to the right of the blue ones, and their contractions are completely disentangled. Then the usual combinatorics yields

$$\langle P(\hat{a})\rangle = {}_{J=0}\langle P(\hat{b})\; e^{-S_{int}[\hat{b}]}\rangle_{\partial=0,\text{ no vac}} \tag{5-117}$$

in self-explanatory notation. Of course this also holds in the quasiassociative version, but the derivation is perhaps less transparent.

[8] associativity helps here.

In general, it is not easy to evaluate these expressions explicitly, because of the coassociators. However the lowest–order corrections $o(h)$ where $q = e^h$ are easy to obtain, using the fact that $\tilde{\phi} = \mathbf{1} + o(h^2)$ for minimal twists (5-15). If we write

$$R_{12} = \mathbf{1} + hr_{12} \ + o(h^2),$$

then

$$\tilde{R}_{12} = R_{12}\sqrt{R_{21}R_{12}}^{-1} = \mathbf{1} + \frac{h}{2}(r_{12} - r_{21}) \ + o(h^2),$$

which allows to find the leading $o(h)$ corrections to the undeformed correlation functions explicitly.

Reality structure. One advantage of this formalism is that the reality structure is naturally induced from the Hilbert space $V_{\mathcal{A}}$, as explained in Section 5.4.1. Using Proposition 5.4.3 and noting that the $a^{K,m}$ are in the contragredient representation of $U(su(2))$, it follows that

$$(\hat{a}^{K,n})^* = \tilde{g}^K_{nm}\hat{a}^{K,m}. \tag{5-118}$$

We shall assume that all the actions are real,

$$S[\hat{\Psi}]^* = S[\hat{\Psi}];$$

this will be verified in the examples below. Moreover, the classical integral defines a real functional on $U(su(2)) \ltimes \mathcal{A}$. Hence we conclude that the correlation functions satisfy

$$\langle P(\hat{a})\rangle^* = \langle P(\hat{a})^*\rangle. \tag{5-119}$$

One can also show that

$$\psi_{I,i}(x)^* = g^I_{ij}\psi_{I,j}(x), \tag{5-120}$$

where g^I_{ij} is normalized such that $g^I_{ij} = (g^I)^{ij}$. Therefore

$$\hat{\Psi}(x)^* = \hat{\Psi}(x), \tag{5-121}$$

using (5-79). This is useful to establish the reality of actions. Of course, one could also consider complex scalar fields. Finally, the correlation functions satisfy the positivity property

$$\langle P(\hat{a})^* P(\hat{a})\rangle \geq 0, \tag{5-122}$$

provided the actions are real. This is a simple consequence of the fact that $P(\hat{a})^* P(\hat{a})$ is a positive operator acting on the left vacuum representation, together with the positivity of the functional integral. It is one of the main merits of the present approach.

5.6 Examples

5.6.1 The free scalar field

Consider the action

$$S_{free}[\Psi] = -\int\limits_{\mathcal{S}^2_{q,N}} \hat{\Psi}(x) \star \Delta\hat{\Psi}(x). \tag{5-123}$$

Here the Laplacian was defined in [36] using a differential calculus as $\Delta = *_H d *_H d$, and satisfies[9]

$$\Delta\psi_{K,n}(x) = \frac{1}{R^2}\,[K]_q[K+1]_q\;\psi_{K,n}(x) \equiv D_K\;\psi_{K,n}(x),$$

where $[K]_q = \frac{q^K - q^{-K}}{q-q^{-1}}$. The basis $\psi_{K,n}(x)$ is normalized as in (5-92). The action is real by (5-121), and can be rewritten as

$$S_{free}[\hat{\Psi}] = -\sum_{K,n} D_K\;\tilde{g}^K_{nm}\hat{a}^{K,m}\hat{a}^{K,n} = -\sum_{K,n} D_K\;(\tilde{g}^K)^{mn}\hat{a}_{K,m}\hat{a}_{K,n},$$

using (5-80). As a first exercise, we calculate the 2–point functions. From (5-112) and (5-113), one finds

$$\begin{aligned}
\langle \hat{a}^K_n \hat{a}^{K'}_{n'}\rangle &= {}_{J=0}\langle \hat{\partial}^K_n\hat{\partial}^{K'}_{n'} Z_{free}[\hat{J}]\rangle_{\partial=0}\\
&= {}_{J=0}\langle \frac{1}{2}\;\hat{\partial}^K_n\hat{\partial}^{K'}_{n'}\;(\sum(\tilde{g}^K)^{rs}\;\hat{j}^K_r D^{-1}_K\hat{j}^K_s)\rangle_{\partial=0}\\
&= D^{-1}_K\;{}_{J=0}\langle\hat{\partial}^K_n\hat{j}^{K'}_{n'}\rangle_{\partial=0} = D^{-1}_K\;\delta^{KK'}\;\tilde{g}^K_{nn'},
\end{aligned}$$

where (5-69) was used in the last line. This result is as expected, and it could also be obtained by using explicitly the definition of the twisted operators $\hat{a}^K$.

The calculation of the 4–point functions is more complicated, since it involves the coassociator. To simplify the notation, we consider the (most complicated) case where all generators a^K have the same spin K, which will be suppressed. The result for the other cases can then be deduced easily. We also omit the prescriptions $(\partial = 0)$ etc. Using first the associative formalism, (5-116) yields

$$\begin{aligned}
\langle \hat{a}_n\hat{a}_m\hat{a}_k\hat{a}_l\rangle &= \langle (D^{-1}\hat{j}_n + \hat{\partial}_n)(D^{-1}\hat{j}_m + \hat{\partial}_m)(D^{-1}\hat{j}_k + \hat{\partial}_k)(D^{-1}\hat{j}_l + \hat{\partial}_l)\rangle\\
&= \langle \hat{\partial}_n(D^{-1}\hat{j}_m + \hat{\partial}_m)(D^{-1}\hat{j}_k + \hat{\partial}_k)D^{-1}\hat{j}_l\rangle\\
&= D^{-2}\;\langle\hat{\partial}_n\hat{j}_m\tilde{g}_{kl}\;+\;\hat{\partial}_n\hat{\partial}_m\hat{j}_k\hat{j}_l\rangle\\
&= D^{-2}\;\langle\tilde{g}_{nm}\tilde{g}_{kl} + \hat{\partial}_n\hat{\partial}_m\hat{j}_k\hat{j}_l\rangle
\end{aligned}$$

To evaluate this, consider

$$\begin{aligned}
\langle\hat{\partial}_n\hat{\partial}_m\hat{j}_k\hat{j}_l\rangle &= \langle\hat{\partial}_n(\tilde{g}_{mk} + \hat{j}_a\hat{\partial}_b\;\mathfrak{R}^{ba}_{mk})\hat{j}_l\rangle\\
&= \tilde{g}_{mk}\tilde{g}_{nl} + \langle\hat{\partial}_n\hat{j}_a\hat{\partial}_b\;\hat{j}_s\pi^s_l(\mathfrak{R}^{ba}_{mk})\rangle\\
&= \tilde{g}_{mk}\tilde{g}_{nl} + \tilde{g}_{na}\tilde{g}_{bs}(\tilde{\phi}_{213}\tilde{R}_{12}\tilde{\phi}^{-1})^{bas}_{mkl}
\end{aligned}$$

[9]it is rescaled from the one in [36] so that its eigenvalues are independent of N.

Collecting the result, we recognize the structure of Wick contractions which are given by the invariant tensor for neighboring indices, but involve the $\tilde{R}$–matrix and the coassociator $\tilde{\phi}$ for "non–planar" diagrams.

To illustrate the quasiassociative approach, we calculate the same 4–point function using the $\star$ product. Then

$$\begin{aligned}\langle a_n \star (a_m \star (a_k \star a_l))\rangle &= \langle \partial_n \star \big((D^{-1}j_m + \partial_m) \star ((D^{-1}j_k + \partial_k) \star D^{-1}j_l)\big)\rangle \\ &= D^{-2}\, \tilde{g}_{nm}\tilde{g}_{kl} + D^{-2}\, \langle \partial_n \star (\partial_m \star (j_k \star j_l))\rangle\end{aligned}$$

using an obvious analog of (5-116). Now

$$\begin{aligned}\langle \partial_n \star (\partial_m \star (j_k \star j_l))\rangle &= \langle \partial_n \star ((\partial_{m'} \star j_{k'}) \star j_{l'})\rangle\, (\tilde{\phi}^{-1})^{m'k'l'}_{mkl} \\ &= \tilde{g}_{nl'}\tilde{g}_{m'k'}(\tilde{\phi}^{-1})^{m'k'l'}_{mkl} + \langle \partial_n \star ((j_{m''} \star \partial_{k''}\tilde{R}^{k''m''}_{m'k'}) \star j_{l'})\rangle\, (\tilde{\phi}^{-1})^{m'k'l'}_{mkl} \\ &= \tilde{g}_{nl}\tilde{g}_{mk} + \langle \partial_n \star (j_{m'} \star (\partial_{k'} \star j_{l'}))\rangle\, (\tilde{\phi}_{213}\tilde{R}_{12}\tilde{\phi}^{-1})^{k'm'l'}_{mkl} \\ &= \tilde{g}_{nl}\tilde{g}_{mk} + \tilde{g}_{nm'}\tilde{g}_{k'l'}\, (\tilde{\phi}_{213}\tilde{R}_{12}\tilde{\phi}^{-1})^{k'm'l'}_{mkl},\end{aligned}$$

in agreement with our previous calculation; here the identity (5-143) was used. As pointed out before, the corrections to order $o(h)$ can now be obtained easily.

5.6.2 Remarks on $N \to \infty$ and ϕ^4 theory.

The above correlators for the free theory are independent of N, as long as the spin of the modes is smaller than N. Therefore one can define the limit $N \to \infty$ in a straightforward way, keeping R constant. In this limit, the algebra of functions on the q–deformed fuzzy sphere becomes

$$\varepsilon^{ij}_k x_i x_j = R\,(q - q^{-1})\,x_k, \qquad g^{ij}x_i x_j = R^2, \tag{5-124}$$

which defines $S^2_{q,N=\infty}$. It has a unique faithful (infinite–dimensional) Hilbert space representation [67].

In an interacting theory, the existence of the limit $N \to \infty$ is of course a highly nontrivial question. Consider for example the ϕ^4 model, with action

$$S[\Psi] = \int_{\mathcal{S}^2_{q,N}} \hat{\Psi}(x)\Delta\hat{\Psi}(x) + m^2\hat{\Psi}(x)^2 + \lambda\hat{\Psi}(x)^4 \; = S_{free} + S_{int}$$

which is real, using (5-121). We want to study the first–order corrections in λ to the 2–point function $\langle \hat{a}^K_i \hat{a}^K_j\rangle$ using (5-117):

$$\langle \hat{a}^K_i \hat{a}^K_j\rangle = {}_{J=0}\langle \hat{b}^K_i\; \hat{b}^K_j\; \Big(1 - \lambda \int_{\mathcal{S}^2_{q,N}} \psi^{I,k}(x)\psi^{J,l}(x)\psi^{L,m}(x)\psi^{M,n}(x)\; \hat{b}^I_k \hat{b}^J_l \hat{b}^L_m \hat{b}^M_n\Big)\rangle_{\partial=0,\text{ no vac}}.$$

We only consider the "leading" planar tadpole diagram. It is given by any contraction of the $\hat{b}^K_i$ and $\hat{b}^K_j$ with $\hat{b}$'s in the interaction term, which does not involve "crossings". All of these

contributions are the same, hence we assume that j is contracted with k and i with l. Then $\hat{b}^L_m$ is contracted with $\hat{b}^M_n$, which gives $D_L^{-1}\tilde{g}^L_{mn}\ \delta^{LM}$. Now $\psi^{L,m}(x)\psi^{L,n}(x)\ \tilde{g}^L_{mn}\ \in \mathcal{S}^2_{q,N}$ is invariant under $U_q(su(2))$ and therefore proportional to the constant function. The numerical factor can be obtained from (5-92):

$$\int \psi^{L,m}(x)\psi^{L,n}(x)\ \tilde{g}^L_{mn} = \tilde{g}_L^{mn}\tilde{g}^L_{mn} = [2L+1]_q = {}_q\mathrm{dim}(V^L).$$

Here V^L denotes the spin L representation of $U_q(su(2))$. Using $\int 1 = 4\pi R^2$, the contribution to $\langle \hat{a}^K_i \hat{a}^K_j\rangle$ is

$$\tilde{g}^K_{il}\tilde{g}^K_{jk}\ \lambda \int \psi^{K,k}(x)\psi^{K,l}(x) \sum_{L=0}^{N}\ D_L^{-1}\ \frac{1}{4\pi R^2}[2L+1]_q = \tilde{g}^K_{ij}\ \frac{\lambda}{4\pi}\sum_{L=0}^{N}\ \frac{[2L+1]_q}{[L]_q[L+1]_q + m^2R^2},$$

up to combinatorial factors of order 1. Unfortunately this diverges linearly in N for $N \to \infty$, whenever $q \neq 1$. This is worse that for $q = 1$, where the divergence is only logarithmic. This is in contrast to a result of [120], which is however in the context of a different concept of (braided) quantum field theory which does not satisfy our requirements in Section 5.5, and hence is not a “smooth deformation” of ordinary QFT. The contributions from the “non–planar” tadpole diagrams are expected to be smaller, because the coassociator $\tilde{\phi}$ as well as $\tilde{R}$ are unitary. At least for scalar field theories, this behavior could be improved by choosing another Laplacian such as $\frac{v-v^{-1}}{q-q^{-1}}$ which has eigenvalues $[2L(L+1)]_q$, where v is the Drinfeld Casimir (5-41). Then all diagrams are convergent as $N \to \infty$. Finally, the case q being a root of unity is more subtle, and we postpone it for future work.

5.6.3 Gauge fields

The quantization of gauge fields $\mathcal{S}^2_{q,N}$ is less clear at present, and we will briefly indicate 2 possibilities. Gauge fields were introduced in Section 2.4.2 as one–forms $B \in \Omega^1_{q,N}$. It is natural to expand the gauge fields in terms of the frame θ^a,

$$B = \sum B_a\theta^a. \tag{5-125}$$

The fact that there are 3 independent one–forms θ^a means that one component is essentially radial and should be considered as a scalar field on the sphere; however, it is impossible to find a (covariant) calculus with “tangential” forms only. Therefore gauge theory on $\mathcal{S}^2_{q,N}$ as presented here is somewhat different from the conventional picture, but may nevertheless be very interesting physically [30].

Actions for gauge theories are expressions in B which involve *no* explicit derivative terms. We recall the simplest examples from Section 2.4.2,

$$S_3 = \int B^3, \quad S_2 = \int B *_H B, \quad S_4 = \int B^2 *_H B^2, \tag{5-126}$$

where $*_H$ is the Hodge star operator. The curvature was defined as $F = B^2 - *_H B$. The meaning of the field B becomes more obvious if it is written in the form

$$B = \Theta + A \tag{5-127}$$

where $\Theta \in \Omega^1_{q,N}$ is the generator of exterior derivatives. While B and Θ become singular in the limit $N \to \infty$, A remains well–defined. In these variables, a more standard form of the actions is recovered, including Yang–Mills

$$S_{YM} := \int F *_H F = \int (dA + A^2) *_H (dA + A^2) \tag{5-128}$$

and Chern–Simons

$$S_{CS} := \frac{1}{3}\int B^3 - \frac{1}{2}\int B *_H B = -\text{const} + \frac{1}{2}\int AdA + \frac{2}{3}A^3 \tag{5-129}$$

terms.

Even though these actions (in particular the prescription "no explicit derivatives") are very convincing and have the correct limit at $q = 1$, the precise meaning of gauge invariance is not clear. In the case $q = 1$, gauge transformations have the form $B_a \to U^{-1} B_a U$ for any unitary matrix U, and actions of the above type are invariant. For $q \neq 1$, the integral is a quantum trace which contains an explicit "weight factor"q^{-H}, breaking this symmetry. There is however another symmetry of the above actions where $U_q(su(2))$ acts on the gauge fields B_a as [36]

$$B_a \to u_1 B_a S u_2 \tag{5-130}$$

or equivalently $B \to u_1 B S u_2$. This can be interpreted as a gauge transformation, leaving the actions invariant for any $u \in U_q(su(2))$ with $\varepsilon_q(u) = 1$, and it is distinct from the rotations of B. There is no obvious extension to a deformed $U(su(N))$ invariance, however. There is yet another $\tilde{U}_q(su(2))$ symmetry, rotating the frames θ^a only, i.e. mixing the components B_a. The rotation of the field B is rather complicated if expressed in terms of the B_a, however.

The significance of all these different symmetries is not clear, and we are not able to preserve them simultaneously at the quantum level. We will therefore indicate two possible quantization schemes, leaving different symmetries manifest.

1) Quantization respecting rotation–invariance. First, we want to preserve the $U_q(su(2))$ symmetry corresponding to rotations of the one–forms $\Omega^1_{q,N}$, which is underlies their algebraic properties [36]. We shall moreover impose the constraint

$$d *_H B = 0,$$

which can be interpreted as gauge fixing. It is invariant under rotations, and removes precisely the null–modes in the Yang–Mills and Chern–Simons terms. We expand the field B into irreducible representations under this action of $U_q(su(2))$:

$$B = \sum_{K,n;\alpha} \Xi^\alpha_{K,n}(x)\ b^{K,n}_\alpha. \tag{5-131}$$

Here $\Xi^{\alpha}_{K,n}(x) \in \Omega^1_{q,N}$ are one–forms which are spin K representation of $U_q(su(2))$ (“vector spherical harmonics”). The multiplicity is now generically 2 because of the constraint, labeled by α.

To quantize this, we can use the same methods as in Section 5.5. One can either define a $\star$ product of the coefficients $b^{K,n}_{\alpha}$ as discussed there, or introduce the operators $\hat{b}^{K,n}_{\alpha}$ acting on a left vacuum representation. Choosing the star product approach to be specific, one can then define correlation functions as

$$\langle P_{\star}(b)\rangle = \frac{1}{\mathcal{N}}\int \Delta B \; e^{-S[B]} P_{\star}(b) \tag{5-132}$$

where ΔB is the integral over all $b^{K,n}_{\alpha}$, write down generating functions etc. This approach has the merit that the remarkable solution $B = \Theta$ of the equation $F = 0$ in [36] survives the quantization, because the corresponding mode is a singlet (so that $\hat{b}^{0,0}_{\alpha} = b^{0,0}_{\alpha}$ is undeformed). Incidentally, observe that the bracketings $\int (BB) *_H (BB)$ and $\int B(B *_H (BB))$ in the star–product approach are equivalent, because of (5-15).

2) Quantization respecting “gauge invariance”. First, notice that there is no need for gauge fixing before quantization even for $q = 1$, since the group of gauge transformations is compact. To preserve the symmetry (5-130) as well as the rotation of the θ^a, we expand B into irreducible representations under these 2 symmetries $U_q(su(2))$ and $\tilde{U}_q(su(2))$:

$$B = \sum_{K,n;a} \psi_{K,n}(x)\theta^a \; \beta^{K,n}_a. \tag{5-133}$$

Now $\beta^{K,n}_a$ is a spin K representation of $\tilde{U}_q(su(2))$ and a spin 1 representation of $U_q(su(2))$. These are independent and commuting symmetries, hence the quantization will involve their respective Drinfeld twists $\tilde{\mathcal{F}}$ and $\mathcal{F}$. In the associative approach of Section 5.5 we would then introduce

$$\hat{\beta}^{K,n}_a = \beta^{K,n'}_{a'} \pi^{a'}_a(\mathcal{F}_1^{-1})\pi^n_{n'}(\tilde{S}\tilde{\mathcal{F}}_1^{-1})\mathcal{F}_2^{-1}\tilde{\mathcal{F}}_2^{-1} \;\in\; (\tilde{U}(su(2)) \otimes U(su(2))) \ltimes \mathcal{A}.$$

To avoid confusion, we have used an explicit matrix notation here. The rest is formally as before, and will be omitted. One drawback of this approach is that the above–mentioned solution $B = \Theta$ is somewhat obscured now: the corresponding mode is part of $\beta^{1,n}_a$, but not easily identified. Moreover, “overall” rotation invariance is not manifest in this quantization.

5.6.4 QFT in $2_q + 1$ dimensions, Fock space

So far, we considered 2–dimensional q–deformed Euclidean field theory. In this section, we will add an extra (commutative) time and define a 2+1–dimensional scalar quantum field theory on $\mathcal{S}^2_{q,N}$ with manifest $\tilde{U}_q(su(2)) \times \mathbb{R}$ symmetry, where $\mathbb{R}$ corresponds to time translations. This will be done using an operator approach, with q–deformed creation and anihilation operators

acting on a Fock space. The purpose is mainly to elucidate the meaning of the Drinfeld twists as "dressing transformations".

We consider real scalar field operators of the form

$$\hat{\Psi}(x,t) = \sum_{K,n} \psi^{K,n}(x)\, \hat{a}_{K,n}(t) \;+\; \psi^{K,n}(x)^*\, \hat{a}^+_{K,n}(t) \tag{5-134}$$

where

$$a^{(+)}_{K,n}(t) = U^{-1}(t)\, a^{(+)}_{K,n}(0)\, U(t) \tag{5-135}$$

for some unitary time–evolution operator $U(t) = e^{-iHt/\hbar}$; we will again put $\hbar = 1$. The Hamilton–operator H acts on some Hilbert space $\mathcal{H}$. We will assume that H is invariant under rotations,

$$[H,u] = 0$$

where $u \in U(su(2))$ is an operator acting on $\mathcal{H}$; recall that as (operator) algebra, $\tilde{U}_q(su(2))$ is the same as $U(su(2))$. Rather than attempting some kind of quantization procedure, we shall assume that

$$\hat{a}^{(+)}_{K,n}(t) = \tilde{\mathcal{F}}_1^{-1} \triangleright a^{(+)}_{K,n}(t)\; \tilde{\mathcal{F}}_2^{-1} \;=\; a^{(+)}_{K,m}(t)\; \pi^m_n(\tilde{\mathcal{F}}_1^{-1})\; \tilde{\mathcal{F}}_2^{-1} \tag{5-136}$$

as in (5-57), where $a^{(+)}_{K,n} = a^{(+)}_{K,n}(0)$ are ordinary creation–and anihilation operators generating a oscillator algebra $\mathcal{A}$,

$$\begin{aligned} [a_{K,n}, a^+_{K',n'}] &= \delta_{KK'}\, (g_c)_{nn'}, \\ [a_{K,n}, a_{K',n'}] &= [a^+_{K,n}, a^+_{K',n'}] = 0 \end{aligned}$$

and act on the usual Fock space[10]

$$\mathcal{H} = \oplus\; (a^+_{K,n} ...\; a^+_{K',n'} |0\rangle). \tag{5-137}$$

$\mathcal{H}$ is in fact a representation of $U(su(2)) \ltimes \mathcal{A}$, and the explicit $U(su(2))$–terms in (5-136) are now understood as operators acting on $\mathcal{H}$. Hence the $\hat{a}^{(+)}_{K,n}(t)$ are some kind of dressed creation–and anihilation operators, whose equal–time commutation relations follow from (5-63), (5-70):

$$\begin{aligned} \hat{a}_{K,n}\; \hat{a}^+_{K',n'} &= \delta_{K,K'}\, g_{nn'} + \hat{a}^+_{K',l'}\; \hat{a}_{K,l}\; \mathfrak{R}^{ll'}_{nn'}, \\ \hat{a}^+_{K,n}\; \hat{a}^+_{K',n'} &= \hat{a}^+_{K',l'}\; \hat{a}^+_{K,l}\; \mathfrak{R}^{ll'}_{nn'}, \\ \hat{a}_{K,n}\; \hat{a}_{K',n'} &= \hat{a}_{K',l'}\; \hat{a}_{K,l}\; \mathfrak{R}^{ll'}_{nn'} \end{aligned}$$

where $\hat{a}^{(+)}_{K,n} = \hat{a}^{(+)}_{K,n}(0)$. The Fock space (5-137) can equivalently be written as

$$\mathcal{H} = \oplus\; \hat{a}^{+K,n} ...\; \hat{a}^{+K',n'} |0\rangle. \tag{5-138}$$

[10] note that this is the same as the "left vacuum representation " of the subalgebra generated by the $a^+_{K,n}$, in the notation of Section 5.4.

Here the main point of our construction of a quantum group covariant field theory is most obvious, namely that a symmetrization postulate has been implemented which restricts the number of states in the Hilbert space as in the undeformed case. This is the meaning of the postulate (4) in the introductory discussion of Section 5.5. One could even exhibit a (trivial) action of the symmetric group S_n on the n–particle space, using the unitary transformation induced by the Drinfeld twist $\mathcal{F}$, as in [122]. Moreover, using (5-120) and an analog of (5-118) it follows that

$$\hat{\Psi}(x,t)^* = \hat{\Psi}(x,t).$$

One can also derive the usual formulas for time–dependent perturbation theory, if we assume that the Hamilton operator has the form

$$H = H_{free} + V$$

where

$$H_{free} = \sum_{K=0}^{N} D_K \ (\tilde{g}^K)^{nm} \ \hat{a}^+_{K,n}\hat{a}_{K,m} = \sum_{K=0}^{N} D_K \ (g^K_c)^{nm} \ a^+_{K,n}a_{K,m}, \tag{5-139}$$

and V may have the form

$$V = \int\limits_{\mathcal{S}^2_{q,N}} \hat{\Psi}(x)\hat{\Psi}(x)...\hat{\Psi}(x).$$

Using (5-71), one can see that

$$[H_{free}, \hat{a}^+_{K,l}] = D_K \ \hat{a}^+_{K,l}$$

and similarly for $\hat{a}_{K,l}$. Therefore the eigenvectors of H_{free} have the form $\hat{a}^{+K,n}...\ \hat{a}^{+K',n'}|0\rangle$ with eigenvalues $(D_K + ... + D_{K'}) \in \mathbb{R}$, and if $V = 0$, then the time evolution is given as usual by

$$\hat{a}^+_{K,n}(t) = e^{-iD_K t/\hbar} \ \hat{a}^+_{K,n}, \qquad \hat{a}_{K,n}(t) = e^{iD_K t/\hbar} \ \hat{a}_{K,n}.$$

One can then go to the interaction picture if $V \neq 0$ and derive the usual formula involving time–ordered products. However one must now keep the time–ordering explicit, and there seems to be no nice formula for contractions of time–ordered products. We shall not pursue this any further here.

The main point here is that the above definitions are entirely within the framework of ordinary quantum mechanics, with a smooth limit $q \to 1$ where the standard quantum field theory on the fuzzy sphere is recovered. Again, one could also consider the limit $N \to \infty$ while keeping q constant. The existence of this limit is far from trivial. Moreover there is nothing special about the space $\mathcal{S}^2_{q,N}$ as opposed to other, perhaps higher–dimensional q–deformed spaces, except the technical simplifications because of the finite number of modes. This shows that there is no obstacle in principle for studying deformations of quantum field theory on such spaces.

5.7 Technical complements to Chapter 5

Proof of Proposition 5.2.1: Assume that $\mathcal{F}$ is minimal, so that (5-15) holds. We must show that it can be chosen such that $\mathcal{F}$ is unitary as well. Define

$$A := \mathcal{F}_{23}(1\otimes\Delta)\mathcal{F}, \qquad B := \mathcal{F}_{12}(\Delta\otimes 1)\mathcal{F},$$

so that $\phi = B^{-1}A$. From (5-15) it follows that $(*\otimes*\otimes*)\phi = \phi^{-1}$, hence $AA^* = BB^*$, and more generally

$$f(AA^*) = f(BB^*)$$

for functions f which are defined by a power series. This also implies that

$$Af(A^*A)A^* = Bf(B^*B)B^*$$

for any such f, hence

$$\phi f(A^*A) = f(B^*B)\phi^{*-1} = f(B^*B)\phi.$$

In particular we can choose $f(x) = \sqrt{x}$ which makes sense because of (5-5), and obtain

$$\sqrt{B^*B}^{-1}\phi\sqrt{A^*A} = \phi. \tag{5-140}$$

On the other hand, the element $T := ((*\otimes*)\mathcal{F})\,\mathcal{F}$ commutes with $\Delta(u)$ because $(*\otimes*)\Delta_q(u) = \Delta_q(u^*)$, and so does $\sqrt{T}$, which is well–defined in $U(su(2))[[h]]$ since $\mathcal{F} = 1 + o(h)$. Moreover, T is symmetric, noting that

$$(*\otimes*)(\mathcal{F}_{21}\mathcal{F}^{-1}) = \mathcal{F}\mathcal{F}_{21}^{-1} \tag{5-141}$$

which follows from the well-known relation $(*\otimes*)\mathcal{R} = \mathcal{R}_{21}$ for $q\in\mathbb{R}$. Therefore T is an admissible gauge transformation, and $\mathcal{F}' := \mathcal{F}\sqrt{T}^{-1}$ is easily seen to be unitary (this argument is due to [129]). In particular, since $\mathcal{F}^*\mathcal{F}$ commutes with $\Delta(u)$, it follows that $A^*A = (\mathcal{F}_{23}^*\mathcal{F}_{23})(1\otimes\Delta)(\mathcal{F}^*\mathcal{F})$ and $B^*B = (\mathcal{F}_{12}^*\mathcal{F}_{12})(\Delta\otimes 1)(\mathcal{F}^*\mathcal{F})$. Looking at the definition (5-13), this means that the left–hand side of (5-140) is the gauge transformation of ϕ under a gauge transformation $\mathcal{F}\to\mathcal{F}' := \mathcal{F}\sqrt{T}^{-1}$, which makes $\mathcal{F}$ unitary. Therefore the coassociator is unchanged under this gauge transformation, hence it remains minimal.

Proof of Lemma 5.3.2: We simply calculate

$$\begin{aligned}
a_1^\dagger\star(a_2\star a_3) &= (a_1^\dagger\star a_2)\star a_3\;\tilde\phi_{123}^{-1}\\
&= (g_{12} + a_2\star a_1^\dagger\;\tilde{\mathcal{R}}_{12})\star a_3\;\tilde\phi_{123}^{-1}\\
&= g_{12}\;a_3\;\tilde\phi_{123}^{-1} + a_2\star(a_1^\dagger\star a_3)\;\tilde\phi_{213}\;\tilde{\mathcal{R}}_{12}\tilde\phi_{123}^{-1}\\
&= g_{12}\;a_3\;\tilde\phi_{123}^{-1} + a_2\star(g_{13} +\; a_3\star a_1^\dagger\;\tilde{\mathcal{R}}_{13})\;\tilde\phi_{213}\;\tilde{\mathcal{R}}_{12}\tilde\phi_{123}^{-1}\\
&= g_{12}\;a_3\;\tilde\phi_{123}^{-1} + a_2 g_{13}\tilde\phi_{213}\;\tilde{\mathcal{R}}_{12}\tilde\phi_{123}^{-1} +\;(a_2\star a_3)\star a_1^\dagger\;\tilde\phi_{231}^{-1}\;\tilde{\mathcal{R}}_{13}\;\tilde\phi_{213}\;\tilde{\mathcal{R}}_{12}\tilde\phi_{123}^{-1}\\
&= g_{12}\;a_3\;\tilde\phi_{123}^{-1} + a_2\;g_{31}\tilde\phi_{231}\tilde{\mathcal{R}}_{1,(23)} + (a_2\star a_3)\star a_1^\dagger\;\tilde{\mathcal{R}}_{1,(23)},
\end{aligned}$$

where (5-39) and $g_{31}\tilde{\mathcal{R}}_{13} = g_{13}$ was used in the last step. Now the first identity (5-48) follows immediately along these lines, omitting the inhomogeneous terms. To see the last one (5-50), observe that

$$g_{12}\tilde{\phi}^{-1}_{123} = (g_c)_{12}\mathcal{F}^{-1}_{1,(23)}\mathcal{F}^{-1}_{23}$$

because $g_{12}\mathcal{F}_{(12),3} = g_{12}$, and similarly

$$g_{31}\tilde{\phi}_{231}\tilde{\mathcal{R}}_{1,(23)} = (g_c)_{13}\mathcal{F}^{-1}_{1,(23)}\mathcal{F}^{-1}_{23}.$$

This implies that

$$\Big(g_{12}\ a_3\ \tilde{\phi}^{-1}_{123} + a_2\ g_{31}\tilde{\phi}_{231}\tilde{\mathcal{R}}_{1,(23)}\Big)P^-_{23} = ((g_c)_{12}\ a_3 + (g_c)_{13}\ a_2)(1-\delta^{23}_{32})\mathcal{F}^{-1}_{1,(23)}\mathcal{F}^{-1}_{23} = 0$$

where we used the fact that the undeformed coproduct is symmetric. The second (5-49) follows as above using

$$g_{31}\tilde{\phi}_{231}\tilde{\mathcal{R}}_{1,(23)}g^{23} = \delta^2_1,$$

or simply from (5-46).

Proof of Proposition 5.4.1: Relation (5-67) follows easily from

$$\pi^j_s(u)(g_c)^{rs} = \pi^r_l(Su)(g_c)^{lj} \tag{5-142}$$

To prove (5-66), consider

$$\begin{aligned} g^{ij}\hat{a}_i\hat{a}_j &= g^{ij}\mathcal{F}^{-1}_1 \triangleright a_i(\mathcal{F}^{-1}_{2,1}\mathcal{F}^{-1}_a)\triangleright a_j\mathcal{F}^{-1}_{2,2}\mathcal{F}^{-1}_b \\ &= a_k a_l g^{ij}\pi^k_i(\mathcal{F}^{-1}_1)\pi^l_j(\mathcal{F}^{-1}_{2,1}\mathcal{F}^{-1}_a)\mathcal{F}^{-1}_{2,2}\mathcal{F}^{-1}_b \\ &= a_k a_l \pi^j_n(\gamma')(g_c)^{in}\pi^k_i(\mathcal{F}^{-1}_1)\pi^l_j(\mathcal{F}^{-1}_{2,1}\mathcal{F}^{-1}_a)\mathcal{F}^{-1}_{2,2}\mathcal{F}^{-1}_b. \end{aligned}$$

Now we use $\pi^k_i(\mathcal{F}^{-1}_1)(g_c)^{in} = (g_c)^{kr}\pi^n_r(S\mathcal{F}^{-1}_1)$, therefore

$$\begin{aligned} g^{ij}\hat{a}_i\hat{a}_j &= a_k a_l(g_c)^{kr}\pi^l_r(\mathcal{F}^{-1}_{2,1}\mathcal{F}^{-1}_a\gamma' S\mathcal{F}^{-1}_1)\mathcal{F}^{-1}_{2,2}\mathcal{F}^{-1}_b \\ &= a_k a_l(g_c)^{kl} \end{aligned}$$

because of (5-20).

Proof of Proposition 5.4.2: (5-69) follows easily from (5-66):

$$\begin{aligned} \hat{\partial}_i(g^{jk}\hat{a}_j\hat{a}_k) &= \hat{\partial}_i((g_c)^{jk}a_j a_k) \\ &= \partial_n\pi^n_i(\mathcal{F}^{-1}_1)\mathcal{F}^{-1}_2((g_c)^{jk}a_j a_k) \\ &= \partial_n\pi^n_i(\mathcal{F}^{-1}_1)((g_c)^{jk}a_j a_k)\mathcal{F}^{-1}_2 \\ &= 2a_n\pi^n_i(\mathcal{F}^{-1}_1)\mathcal{F}^{-1}_2 + ((g_c)^{jk}a_j a_k)\partial_n\pi^n_i(\mathcal{F}^{-1}_1)\mathcal{F}^{-1}_2 \\ &= 2\hat{a}_i + (g^{jk}\hat{a}_j\hat{a}_k)\hat{\partial}_i, \end{aligned}$$

as claimed. Next, consider

$$\begin{aligned}\hat{\partial}_i \hat{a}_j &= \partial_n \pi^n_i(\mathcal{F}_1^{-1}) a_l \pi^l_j(\mathcal{F}_{2,1}^{-1}\mathcal{F}_a^{-1})\mathcal{F}_{2,2}^{-1}\mathcal{F}_b^{-1} \\ &= (g_c)_{nl}\ \pi^n_i(\mathcal{F}_1^{-1})\pi^l_j(\mathcal{F}_{2,1}^{-1}\mathcal{F}_a^{-1})\mathcal{F}_{2,2}^{-1}\mathcal{F}_b^{-1} + a_l\pi^l_j(\mathcal{F}_{2,1}^{-1}\mathcal{F}_a^{-1})\partial_n\pi^n_i(\mathcal{F}_1^{-1})\mathcal{F}_{2,2}^{-1}\mathcal{F}_b^{-1}.\end{aligned}$$

The second term becomes $\hat{a}_k\hat{\partial}_l\ \mathfrak{R}^{lk}_{ij}$ as in (5-63), and the first is

$$\begin{aligned}(g_c)_{nl}\ \pi^n_i(\mathcal{F}_1^{-1})\pi^l_j(\mathcal{F}_{2,1}^{-1}\mathcal{F}_a^{-1})\mathcal{F}_{2,2}^{-1}\mathcal{F}_b^{-1} &= \pi^t_l(S\mathcal{F}_1^{-1})(g_c)_{ti}\pi^l_j(\mathcal{F}_{2,1}^{-1}\mathcal{F}_a^{-1})\mathcal{F}_{2,2}^{-1}\mathcal{F}_b^{-1} \\ &= (g_c)_{ti}\ \pi^t_j(S\mathcal{F}_1^{-1}\mathcal{F}_{2,1}^{-1}\mathcal{F}_a^{-1})\mathcal{F}_{2,2}^{-1}\mathcal{F}_b^{-1} \\ &= (g_c)_{ti}\ \pi^t_j(\gamma) = (g_c)_{ti}\ \pi^t_l(S\mathcal{F}_1^{-1})\pi^l_j(\mathcal{F}_2^{-1}) \\ &= (g_c)_{tl}\ \pi^t_i(\mathcal{F}_1^{-1})\pi^l_j(\mathcal{F}_2^{-1}) = g_{ij} \qquad \text{(5-143)}\end{aligned}$$

using (5-19).

Proof of Proposition 5.4.3: Since π is a unitary representation, we have

$$\begin{aligned}\hat{a}_i^* &= \mathcal{F}_2 a_j^* \pi^i_j(\mathcal{F}_1) = \mathcal{F}_2 a_k (g_c)^{kj}\pi^i_j(\mathcal{F}_1) \\ &= \mathcal{F}_2 a_k (g_c)^{ni}\pi^k_n(S\mathcal{F}_1) \\ &= a_l\pi^l_k(\mathcal{F}_{2,1})(g_c)^{in}\pi^k_n(S\mathcal{F}_1)\mathcal{F}_{2,2} \\ &= a_l\pi^l_k(\mathcal{F}_{2,1}S\mathcal{F}_1)\mathcal{F}_{2,2}g^{it}\pi^k_t(\gamma'^{-1}) \\ &= a_l\pi^l_t(\mathcal{F}_{2,1}S\mathcal{F}_1\gamma'^{-1})\mathcal{F}_{2,2}g^{it} \\ &= a_l\pi^l_t(\mathcal{F}_1^{-1})\mathcal{F}_2^{-1}g^{it} \\ &= \hat{a}_t g^{it}\end{aligned}$$

where (5-18) was essential.

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